QUANTUM DIMENSIONS AND THEIR NON-ARCHIMEDEAN DEGENERATIONS

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ABSTRACT. We derive explicit dimension formulas for irreducible $\mathcal{M}_{\mathbb{F}}$-spherical $\mathcal{K}_{\mathbb{F}}$-representations where $\mathcal{K}_{\mathbb{F}}$ is the maximal compact subgroup of the general linear group $\mathrm{GL}_d(\mathbb{F})$ over a local field $\mathbb{F}$ and $\mathcal{M}_{\mathbb{F}}$ is a closed subgroup of $\mathcal{K}_{\mathbb{F}}$ such that $\mathcal{K}_{\mathbb{F}}/\mathcal{M}_{\mathbb{F}}$ realizes the Grassmannian of n-dimensional $\mathbb{F}$-subspaces of $\mathbb{F}^d$. We explore the fact that $(\mathcal{K}_{\mathbb{F}}, \mathcal{M}_{\mathbb{F}})$ is a Gelfand pair whose associated zonal spherical functions identify with various degenerations of the multivariable little q-Jacobi polynomials. As a result, we are led to consider generalized dimensions defined in terms of evaluations and quadratic norms of multivariable little q-Jacobi polynomials, which interpolate between the various classical dimensions. The generalized dimensions themselves are shown to have representation theoretic interpretations as the quantum dimensions of irreducible spherical quantum representations associated to quantum complex Grassmannians.

1. INTRODUCTION

Let $\mathbb{F}$ be a local field and let $\mathcal{K}_{\mathbb{F}}$ be the maximal compact subgroup of $\mathrm{GL}_d(\mathbb{F})$. That is,

$$(1.1) \qquad \mathcal{K}_{\mathbb{F}} = \begin{cases} \mathrm{O}(d) & \text{if } \mathbb{F} = \mathbb{R}, \\ \mathrm{U}(d) & \text{if } \mathbb{F} = \mathbb{C}, \\ \mathrm{GL}_d(\mathcal{O}) & \text{if } \mathbb{F} \text{ non-Archimedean}, \end{cases}$$

where in the latter case $\mathcal{O}$ denotes the ring of integers of $\mathbb{F}$. Let $\mathcal{G}(n, d; \mathbb{F})$ be the Grassmannian of n-dimensional subspaces of a fixed d-dimensional vector space over $\mathbb{F}$. Throughout we assume that $n \leq d/2$. The group $\mathcal{K}_{\mathbb{F}}$ acts transitively on $\mathcal{G}(n, d; \mathbb{F})$. Letting $\mathcal{M}_{\mathbb{F}}$ be a stabilizer of a point we may identify $\mathcal{G}(n, d; \mathbb{F})$ with $\mathcal{K}_{\mathbb{F}}/\mathcal{M}_{\mathbb{F}}$. The pair $(\mathcal{K}_{\mathbb{F}}, \mathcal{M}_{\mathbb{F}})$ is a Gelfand pair, in the sense that each (continuous, complex) irreducible $\mathcal{K}_{\mathbb{F}}$-representation has a subspace of $\mathcal{M}_{\mathbb{F}}$-fixed vectors which is at most one-dimensional. The irreducible representations having a one-dimensional subspace of $\mathcal{M}_{\mathbb{F}}$-fixed vectors are called $\mathcal{M}_{\mathbb{F}}$-spherical. In this paper we derive explicit dimension formulas for the irreducible $\mathcal{M}_{\mathbb{F}}$-spherical $\mathcal{K}_{\mathbb{F}}$-representations.

For all local fields $\mathbb{F}$ the equivalence classes of the irreducible $\mathcal{M}_{\mathbb{F}}$-spherical $\mathcal{K}_{\mathbb{F}}$-representations are naturally parameterized by the set Λ_n of partitions of at most n parts

(see e.g. [12] and [5] for $\mathbb{F}$ Archimedean, and [2] for $\mathbb{F}$ non-Archimedean). This is quite remarkable, taking into consideration that the representation theory of $\mathcal{K}_{\mathbb{F}}$ for $\mathbb{F}$ non-Archimedean is of a completely different nature compared to the representation theory for $\mathbb{F}$ Archimedean. In each case there is a natural explicit labeling of the irreducible $\mathcal{M}_{\mathbb{F}}$-spherical $\mathcal{K}_{\mathbb{F}}$-representations $V_\lambda^{\mathbb{F}}$ by partitions $\lambda \in \Lambda_n$ which we shall describe in detail.

The zonal spherical functions $\varphi_\lambda^{\mathbb{F}}$ associated to $V_\lambda^{\mathbb{F}}$ ($\lambda \in \Lambda_n$) form a linear basis of the Hecke algebra of $\mathcal{M}_{\mathbb{F}}$-biinvariant representative functions on $\mathcal{K}_{\mathbb{F}}$. The starting point for the derivation of the explicit dimension formulas for the irreducible $\mathcal{M}_{\mathbb{F}}$-spherical $\mathcal{K}_{\mathbb{F}}$-representations is the representation theoretic dimension formula

$$(1.2) \qquad \mathrm{Dim}_{\mathbb{C}}\big(V_\lambda^{\mathbb{F}}\big) = \frac{\big(\varphi_\lambda^{\mathbb{F}}(e)\big)^2}{\int_{\mathcal{K}_{\mathbb{F}}} |\varphi_\lambda^{\mathbb{F}}(g)|^2 dg}, \qquad \lambda \in \Lambda_n$$

where $e \in \mathcal{K}_{\mathbb{F}}$ is the unit element and integration is with respect to the normalized Haar measure on $\mathcal{K}_{\mathbb{F}}$. The second key ingredient is the fact that the zonal spherical functions $\varphi_\lambda^{\mathbb{F}}$ for the different local fields $\mathbb{F}$ relate to various degenerations of the multivariable little q-Jacobi polynomials from [26], see [12], [5] and [23]. As a result, we are led to consider generalized dimensions defined in terms of evaluations and quadratic norms of multivariable little q-Jacobi polynomials, which interpolate between the various classical dimensions.

The multivariable little q-Jacobi polynomials $P_\lambda^L(z)$ ($\lambda \in \Lambda_n$) are symmetric polynomials in n variables $z = (z_1, \ldots, z_n)$ depending on four auxiliary parameters, which arise as degenerations of the Macdonald-Koornwinder polynomials [15]. They form an orthogonal system with respect to an explicit discrete probability orthogonality measure defined in terms of an iterated Jackson integral (see [26]), which induces a norm $\|\cdot\|_L$ on the symmetric polynomials in z. The generalized dimensions referred to in the previous paragraph are

$$(1.3) \qquad d(\lambda) = \frac{P_\lambda^L(0)^2}{\|P_\lambda^L\|_L^2}, \qquad \lambda \in \Lambda_n,$$

compare with the representation theoretic dimension formula (1.2).

We show that the dimension of $V_\lambda^{\mathbb{F}}$ can be obtained as a classical ($q = 1$) degeneration of $d(\lambda)$ for $\mathbb{F}$ Archimedean, and as a p-adic ($q = 0$) degeneration of $d(\lambda)$ for $\mathbb{F}$ non-Archimedean. For special values of the four auxiliary parameters of the multivariable little q-Jacobi polynomials we show that the generalized dimensions $d(\lambda)$ ($\lambda \in \Lambda_n$) themselves have a representation theoretic interpretation as the quantum dimensions of irreducible spherical quantum representations associated to the one-parameter family of quantum complex Grassmannians from [20] and [5]. In this case the key ingredient is the identification in [20] and [5] of the associated quantum zonal spherical functions with Macdonald-Koornwinder polynomials, as well as with degenerations of the Macdonald-Koornwinder polynomials known as multivariable big and little q-Jacobi polynomials.

Cherednik's double affine Hecke algebra techniques have led to explicit evaluation formulas and to explicit quadratic norm evaluations for the Macdonald-Koornwinder polynomials in [24] (see also [3] and [28]). The quadratic norms $\|P_\lambda\|_L^2$ of the multivariable little q-Jacobi polynomials (as well as for multivariable big q-Jacobi polynomials) have been explicitly evaluated in [27] by degenerating the quadratic norm evaluations of the

Macdonald-Koornwinder polynomials. In exactly the same way we derive in this paper evaluation formulas for the multivariable big and little q-Jacobi polynomials. This leads to explicit expressions for the interpolating, generalized dimensions $d(\lambda)$ ($\lambda \in \Lambda_n$). Explicit (quantum) dimension formulas for the irreducible spherical (quantum) representations associated to the various (quantum) Grassmannians are subsequently derived by degenerating the explicit expression for $d(\lambda)$ ($\lambda \in \Lambda_n$).

The contents of the paper is as follows. In Section 2 we introduce the multivariable little q-Jacobi polynomials and the associated generalized dimensions. In Section 3 we discuss the representation theory of general Compact Quantum Group (CQG) algebras from [4] and [7]. CQG algebras form the natural setting to capture the harmonic analytic structures of the various (quantum) Grassmannians under consideration in this paper.

In Section 4 we specialize the general representation theoretic setup from Section 3 to the classical CQG algebra of representative functions on the compact group $\mathcal{K}_{\mathbb{F}}$. We discuss the harmonic analytic implications for the associated Grassmannian $\mathcal{K}_{\mathbb{F}}/\mathcal{M}_{\mathbb{F}}$. We pay close attention to the specific parametrization of the irreducible $\mathcal{M}_{\mathbb{F}}$-spherical $\mathcal{K}_{\mathbb{F}}$-representations by Λ_n. We state the explicit dimension formulas, which are new in case of non-Archimedean local fields $\mathbb{F}$ (see Theorem 4.5). Although the results are essentially classical for Archimedean local fields $\mathbb{F}$ (see e.g. [12], [5], [30] and [23]), we have decided to give a detailed exposition in case of $\mathbb{F} = \mathbb{C}$ in order to highlight the similarities and differences to the results for the Grassmannians over non-Archimedean local fields $\mathbb{F}$, as well as to the results for the quantum complex Grassmannians in the subsequent sections.

In Section 5 we specialize the setup from Section 3 to the CQG algebra of quantized representative functions on $\mathcal{K}_{\mathbb{C}} = \mathrm{U}(n)$, and we discuss the associated harmonic analysis on the standard quantum complex Grassmannian. In particular, we state explicit quantum dimension formulas for the associated irreducible spherical quantum representations.

In Section 6 we introduce the Macdonald-Koornwinder polynomials and the multivariable big and little q-Jacobi polynomials. We derive explicit evaluation formulas for the multivariable big and little q-Jacobi polynomials. Subsequently we derive in Section 7 explicit expressions for the generalized dimensions as well as for their p-adic ($q = 0$) degenerations.

In Section 8 we discuss the present approach for the one-parameter family of quantum complex Grassmannians from [20] and [5]. Following closely the analysis of Noumi [19] for other examples of quantum symmetric spaces, we express the quantum dimensions of the associated irreducible spherical quantum representations in terms of evaluations and quadratic norms of Macdonald-Koornwinder polynomials, as well as of multivariable big and little q-Jacobi polynomials. In particular, we obtain the representation theoretic interpretation of the generalized dimensions for special parameters as quantum dimensions of irreducible spherical quantum representations. In this context we remark that a similar approach can be followed for quantum real Grassmannians using the harmonic analytic results from [30] and [17], but we do not pursue it in this paper.

Notations and conventions. Representations of topological groups are complex and are required to be continuous. We set

$$\mathbb{E} = \mathbb{C}(a, b, c, d, q, t) \tag{1.4}$$

for the field of rational functions in six indeterminates a, b, c, d, q and t. We set Λ_n for the partitions of at most n parts. It is the cone

$$\Lambda_n = \bigoplus_{r=1}^{n} \mathbb{Z}_+ \omega_r$$

in $\mathbb{Z}^n$, where $\omega_r = (1^r, 0^{n-r}) \in \Lambda_n$ is the fundamental partition consisting of r ones and $n - r$ zeros. We write $\omega_0 = (0^n) \in \Lambda_n$ for the zero partition. For a partition $\lambda \in \Lambda_n$ we denote λ' for the conjugate partition and we set $\lambda'_0 = n$. In addition we write $|\lambda|$ for the weight of λ, $l(\lambda) = \lambda'_1$ for the length of λ, and $\partial \lambda' = (\lambda'_j - \lambda'_{j+1})_{j \geq 0}$ for the positive integral differences of the conjugate partition (whose sum equals n). For n-vectors u, v, w we set $uv^w = (u_1 v_1^{w_1}, \ldots, u_n v_n^{w_n})$. We occasionally use this notation when u or v are scalars, in which we use the corresponding n-vectors having all entries equal to the given scalar. We recall some standard notations from basic hypergeometric series (see e.g. [8]). For $j \in \mathbb{Z}_+$ we set $(a; q)_j = (1 - a)(1 - aq) \cdots (1 - aq^{j-1})$ (empty product is one) for the q-shifted factorial. We write $(a_1, \ldots, a_m; q)_j = \prod_{s=1}^{m} (a_s; q)_j$ for products of q-shifted factorials. For $m, l, l_1, \ldots, l_k \in \mathbb{Z}_+$ with $l \leq m$, we use the notations

$$[m] = \frac{1 - q^m}{1 - q}, \qquad\qquad [m]_q! = \prod_{j=1}^{m} [j]_q,$$

$$\begin{bmatrix} m \\ l_1 \ l_2 \ \cdots \ l_k \end{bmatrix}_q = \frac{[m]_q!}{[l_1]_q! \cdots [l_k]_q!}, \qquad \binom{m}{l}_q = \begin{bmatrix} m \\ l \ m - l \end{bmatrix}_q.$$

In terms of q-shifted factorials, the q-factorial can be expressed as

$$[m]_q! = q^{\frac{1}{2}m(m-1)} \frac{(q^{-1}; q^{-1})_m}{(1 - q^{-1})^m}. \tag{1.5}$$

We use standard notations for the basic hypergeometric series $_{r+1}\phi_r$, see e.g. [8].

2. MULTIVARIABLE LITTLE q-JACOBI POLYNOMIALS

Important in our approach is the fact that (quantum) dimensions of irreducible spherical representations associated to quantum, real/complex and p-adic Grassmannians are degenerations of a generalized dimension formula involving multivariable little q-Jacobi polynomials, which we now introduce. We work over the field $\mathbb{E}$ (see (1.4)), although in the present situation c and d are dummy parameters and could just as well be omitted (in contrast to the Macdonald-Koornwinder and multivariable big q-Jacobi case as treated in Section 6).

The symmetric group S_n in n letters acts on $\mathbb{Z}_+^n$ by permuting the coordinates. The corresponding fundamental domain is the cone Λ_n of partitions with at most n parts. The

symmetric group S_n acts on $\mathbb{E}[z] = \mathbb{E}[z_1, \ldots, z_n]$ by permuting the independent variables $z_1, \ldots, z_n$. We denote $\mathbb{E}[z]^S$ for the subalgebra of S_n-invariant polynomials in $\mathbb{E}[z]$. An $\mathbb{E}$-basis of $\mathbb{E}[z]^S$ is formed by the symmetric monomials $\widetilde{m}_\lambda(z) = \sum_{\mu \in S_n \lambda} z^\mu$ ($\lambda \in \Lambda_n$). The monic multivariable little q-Jacobi polynomial $P_\lambda^L(z) = P_\lambda^L(z; a, b; q, t) \in \mathbb{E}[z]^S$ of degree $\lambda \in \Lambda_n$ from [26] is of the form

$$(2.1) \qquad P_\lambda^L(z) = \widetilde{m}_\lambda(z) + \sum_{\mu \in \Lambda_n : \mu < \lambda} c_{\lambda,\mu}^L \widetilde{m}_\mu(z)$$

for certain coefficients $c_{\lambda,\mu}^L = c_{\lambda,\mu}^L(a, b; q, t) \in \mathbb{E}$. The multivariable little q-Jacobi polynomials can be characterized as solution of a second-order difference equation, or in terms of orthogonality relations defined with respect to an explicit discrete probability orthogonality measure. The characterization in terms of the difference equation is as follows. For $j = 1, \ldots, n$ we write T_j for the multiplicative q-shift in z_j,

$$(T_j f)(z) = f(z_1, \ldots, z_{j-1}, q z_j, z_{j+1}, \ldots, z_n), \qquad f \in \mathbb{E}[z^{\pm 1}].$$

The characterizing difference equation then reads $D_L P_\lambda^L = E_\lambda^{qJ} P_\lambda^L$ where

$$D_L = \sum_{j=1}^n \big(\phi_{L,j}^+(z)(T_j - \mathrm{Id}) + \phi_{L,j}^-(z)(T_j^{-1} - \mathrm{Id}) \big),$$

$$\phi_{L,j}^+(z) = q t^{n-1} a \big(b - \frac{1}{q z_j} \big) \prod_{l \neq j} \frac{z_l - t z_j}{z_l - z_j}, \qquad \phi_{L,j}^-(z) = \big(1 - \frac{1}{z_j} \big) \prod_{l \neq j} \frac{z_j - t z_l}{z_j - z_l}$$

and with eigenvalue

$$E_\lambda^{qJ} = \sum_{j=1}^n \big(q a b t^{2n-j-1}(q^{\lambda_j} - 1) + t^{j-1}(q^{-\lambda_j} - 1) \big).$$

An algebraic formulation of the orthogonality relations of the multivariable little q-Jacobi polynomials is as follows. Define an $\mathbb{E}$-linear functional

$$h_L = h_L^{a,b;q,t} : \mathbb{E}[z]^S \to \mathbb{E}$$

by requiring

$$h_L\big(P_\lambda^L\big) = \begin{cases} 1, & \text{if } \lambda = 0, \\ 0, & \text{if } \lambda \neq 0. \end{cases}$$

Define a form $\langle \cdot, \cdot \rangle_L = \langle \cdot, \cdot \rangle_L^{a,b;q,t}$ on $\mathbb{E}[z]^S$ by

$$(2.2) \qquad \langle p_1, p_2 \rangle_L = h_L\big(p_1(z)\overline{p_2(z)}\big), \qquad p_1, p_2 \in \mathbb{E}[z]^S,$$

where $p(z) \mapsto \overline{p(z)}$ stands for the anti-linear algebra involution on $\mathbb{E}[z]$ which extends complex conjugation on $\mathbb{C}$ by requiring that the parameters a, b, c, d, q, t and the variables z_j to be formally real, e.g. $\overline{a} = a$ and $\overline{z_j} = z_j$. The orthogonality relations now take the form

$$(2.3) \qquad \langle P_\lambda^L, P_\mu^L \rangle_L = N_L(\lambda) \delta_{\lambda,\mu}, \qquad \lambda, \mu \in \Lambda_n$$

for suitable explicit quadratic norms $N_L(\lambda) = N_L(\lambda; a, b; q, t) \in \mathbb{E}$, which are recalled in Subsection 6.2.

The expressions

$$(2.4) \qquad D_q(\lambda; a, b; t) = \frac{\left(P_\lambda^L(0; q^{-1}a, q^{-1}b; q, t)\right)^2}{N_L(\lambda; q^{-1}a, q^{-1}b; q, t)} \in \mathbb{E}, \qquad \lambda \in \Lambda_n$$

play a crucial role in the sequel. Note that $D_q(\lambda; a, b; t)$ is the generalized dimension $d(\lambda)$ from the introduction (see (1.3)), up to a slight reparametrization of the four auxiliary parameters a, b, q and t (which we justify in a moment).

We show that specializations of the parameters (a, b, t) in $D_q(\lambda)$ give the quantum dimensions of spherical irreducible representations associated to quantum complex Grassmannians. In addition we show that suitable degenerations in (a, b, q, t) lead to the dimensions of spherical irreducible representations associated to complex/real Grassmannians (in which case the four parameters degenerate to one, but with different exponents), and to dimensions of spherical irreducible representations associated to Grassmannians over non-Archimedean local fields (in which case q degenerates to zero while the remaining three parameters (a, b, t) are specialized to q-independent values which are explicitly given in terms of the cardinality of the residue field). The latter case thus does not involve degenerating the parameters (a, b, t), which is the reason to use the multivariable little q-Jacobi polynomials with parameters $(q^{-1}a, q^{-1}b, t)$ in the definition (2.4) of the generalized dimensions.

We derive evaluation formulas for P_λ^L in Section 6, which thus lead to explicit evaluations of $D_q(\lambda) = D_q(\lambda; a, b; t)$ and to explicit evaluations of the (quantum) dimensions for the irreducible spherical irreducible representations associated to different types of (quantum) Grassmannians.

3. CQG ALGEBRAS AND QUANTUM DIMENSIONS

The category of Compact Quantum Group algebras (CQG) is a full subcategory of the category of Hopf $*$-algebras that includes the standard noncommutative deformations of algebras of representative functions on compact Lie groups. In particular, CQG algebras are Hopf $*$-algebras with positive definite Haar functionals which have natural analogs of the Peter-Weyl decomposition and the Schur orthogonality relations. This leads to the notion of the quantum dimension of an irreducible, finite dimensional comodule over a given CQG algebra. One of the main aims of the paper is to explicitly compute quantum dimensions of irreducible comodules which arise in the context of (quantum) Grassmannians. Following [4], we recall in this section the definition of a CQG algebra and its basic properties. We furthermore discuss the notion of a quantum Gelfand pair in the context of CQG algebras (cf. [7]), and the associated quantum zonal spherical functions.

Let A be a Hopf $*$-algebra with comultiplication Δ, counit ϵ and antipode S. A linear functional $h : A \to \mathbb{C}$ is called a normalized Haar functional if $h(1) = 1$ and

$$\left(h \otimes \mathrm{Id}_A\right)\left(\Delta(a)\right) = h(a)1_A = \left(\mathrm{Id}_A \otimes h\right)\left(\Delta(a)\right), \qquad a \in A.$$

A Haar functional h on A is called positive definite if $h(a^*a) > 0$ for all $0 \neq a \in A$. There are different equivalent definitions of Dijkhuizen's and Koornwinder's [4] notion of a CQG algebra, the following is a convenient starting point for our purposes.

Definition 3.1. *We call the Hopf $*$-algebra A a Compact Quantum Group (CQG) algebra if there exists a positive definite Haar functional on A.*

The positive definite Haar functional on a CQG algebra A is unique. Several concrete examples of CQG algebras (such as the algebra of representative functions on a compact group, the basic commutative example) will be discussed in detail in the following two sections.

Let A be a CQG algebra with normalized, positive definite Haar functional h. The following results are all from [4]. Any finite-dimensional right A-comodule is unitarizable, hence completely reducible. We denote Σ for the set of equivalence classes of finite dimensional, (unitary) irreducible right A-comodules. For $\alpha \in \Sigma$ let $\pi_\alpha : L_\alpha \to L_\alpha \otimes A$ be a representative of the corresponding irreducible, finite dimensional A-comodule. Let $\pi_\alpha^* : L_\alpha^* \to L_\alpha^* \otimes A$ be the corresponding dual right A-comodule, defined by requiring

$$\big(\pi_\alpha^*(\phi)\big)\big(v \otimes 1_A\big) = \big(\phi \otimes S\big)\big(\pi_\alpha(v)\big), \qquad v \in L_\alpha, \ \phi \in L_\alpha^*,$$

where 1_A denotes the unit element of A. Then $\{L_\alpha^* \otimes L_\alpha \,|\, \alpha \in \Sigma\}$ is a complete set of representatives of the irreducible, finite dimensional $A \otimes A$-comodules.

Consider A as a right $A \otimes A$-comodule by

$$R(a) = \sum_{(a)} a_{(2)} \otimes S(a_{(1)}) \otimes a_{(3)}, \qquad a \in A,$$

where we use the standard Sweedler notation for (iterated) applications of the comultiplication. The first key property of CQG algebras is the irreducible decomposition

$$A \simeq \bigoplus_{\alpha \in \Sigma} L_\alpha^* \otimes L_\alpha$$

of A as right $A \otimes A$-comodule (the Peter-Weyl Theorem for CQG algebras). Concretely, the isomorphism is realized by mapping $\phi \otimes v \in L_\alpha^* \otimes L_\alpha$ to $(\phi \otimes \mathrm{Id}_A)(\pi_\alpha(v)) \in A$. We denote A_h for the Hilbert space completion of A with respect to the pre-Hilbert structure

$$\langle a, b \rangle_h := h(b^*a).$$

Let (π, V) be a finite dimensional, irreducible, right A-comodule. Consider the canonical linear isomorphism $\varphi : V \to V^{**}$ defined by $\varphi(v)f := f(v)$ for $f \in V^*$ and $v \in V$. We turn V into a finite dimensional Hilbert space such that (π, V) is a unitary right A-comodule (such scalar product is unique up to a strictly positive scalar multiple). Since the double-dual right A-comodule (π^{**}, V^{**}) is equivalent to (π, V), there exists a unique positive definite, linear isomorphism $F = F_V : V \to V$ such that

$$\mathrm{Tr}_V(F) = \mathrm{Tr}_V(F^{-1}) > 0$$

and such that $\varphi \circ F : (\pi, V) \to (\pi^{**}, V^{**})$ is an intertwiner of right A-comodules (see [4, Prop. 3.6]).

Definition 3.2. *Let A be a CQG algebra, (π, V) a finite dimensional, irreducible right A-comodule and $F : V \to V$ the linear isomorphism as defined above. Then*

$$\mathrm{Dim}_A(V) := \mathrm{Tr}_V(F)$$

is called the quantum dimension of the right A-comodule (π, V).

A second key property of CQG algebras is the generalized Schur orthogonality relations [4, Prop. 3.4 & 3.5], which we now proceed to recall.

Proposition 3.3. *Let A be a CQG algebra. Let (π, V) and (π', V') be finite dimensional, irreducible, unitary right A-comodules with associated scalar products $\langle \cdot, \cdot \rangle_\pi$ and $\langle \cdot, \cdot \rangle_{\pi'}$.*

(a) *If $\pi \nsimeq \pi'$, then*

$$\langle \langle \pi(u), v \rangle_\pi, \langle \pi'(u'), v' \rangle_{\pi'} \rangle_h = 0$$

   for $u, v \in V$ and $u', v' \in V'$.

(b) *For $u_1, u_2, v_1, v_2 \in V$ we have*

$$\langle \langle \pi(u_1), v_1 \rangle_\pi, \langle \pi(u_2), v_2 \rangle_\pi \rangle_h = \frac{\langle u_1, u_2 \rangle_\pi \langle F_V^{-1} v_2, v_1 \rangle_\pi}{\mathrm{Dim}_A(V)}.$$

Let A be a CQG algebra. A quantum subgroup of A is a pair (B, p) with B a Hopf $*$-algebra and $p : A \to B$ a surjective Hopf $*$-algebra homomorphism. In this situation, we write

$$A_B = \{a \in A \,|\, (\mathrm{Id}_A \otimes p)(\Delta(a)) = a \otimes 1_B\}$$

for the $*$-subalgebra of right B-invariant elements in A. The $*$-subalgebra $A_B \subset A$ is a right A-comodule under the right regular co-action

$$\rho_B(a) = \sum_{(a)} a_{(2)} \otimes S(a_{(1)}), \qquad a \in A_B.$$

By the Peter-Weyl Theorem for A, the right A-comodule A_B decomposes into irreducibles as

(3.1) $$A_B \simeq \bigoplus_{\alpha \in \Sigma} L_\alpha^{* \ \mathrm{Dim}_{\mathbb{C}}(L_\alpha^B)},$$

where for a right A-comodule (π, V),

$$V^B = \{v \in V \,|\, (1 \otimes p)(\pi(v)) = v \otimes 1_B\}$$

is the space of B-invariant elements in V.

Definition 3.4. *A CQG algebra A with quantum subgroup (B, p) is called a quantum Gelfand pair if*

$$\mathrm{Dim}_{\mathbb{C}}(L_\alpha^B) \leq 1 \qquad \forall \, \alpha \in \Sigma,$$

(compare with [7]).

We fix a quantum Gelfand pair (A, B). We call a finite dimensional, irreducible A-comodule B-spherical if it has nonzero B-invariant vectors. We denote $\Sigma_B \subset \Sigma$ for the corresponding subset of equivalence classes of the B-spherical, irreducible, finite dimensional right A-comodules.

In the various examples associated to (quantum) Grassmannians which we discuss in the following sections, we will identify Σ_B with a certain fixed set of partitions. An important, but case-by-case different, aspect is the concrete characterization of the spherical irreducibles in terms of their parameterizing set Σ_B. To maintain uniform notations as much as possible, it will therefore be convenient to denote irreducible comodules labeled by Σ_B by new symbols $\{V_\alpha \mid \alpha \in \Sigma_B\}$.

We write $\langle \cdot, \cdot \rangle_\alpha$ for a scalar product on V_α that turns V_α into a unitary right A-comodule. Note that Σ_B is invariant under $\circ$, were $\circ$ is the involution on Σ such that $V_\alpha^* \simeq V_{\alpha^\circ}$ for all $\alpha \in \Sigma$. In fact, if we fix a nonzero vector $v_\alpha \in V_\alpha^B$ ($\alpha \in \Sigma_B$), then $\langle \cdot, v_\alpha \rangle_\alpha \in V_\alpha^{*B}$. Combined with (3.1), we obtain the multiplicity free irreducible decomposition

$$(3.2) \qquad\qquad A_B = \bigoplus_{\alpha \in \Sigma_B} V_\alpha$$

as right A-comodules.

From the present perspective, the natural definition of the quantum zonal spherical functions would be

$$\widetilde{\varphi}_\alpha = \langle \pi_\alpha(v_\alpha), v_\alpha \rangle_\alpha, \qquad \forall\, \alpha \in \Sigma_B,$$

since they form a $\mathbb{C}$-basis of the $*$-algebra

$${}_B A_B = \{ a \in A \mid (\mathrm{Id}_A \otimes p \otimes p) R(a) = a \otimes 1_B \otimes 1_B \}$$

of B-biinvariant elements in A. It turns out though that the following slightly modified definition of the quantum zonal spherical functions is a better choice.

Definition 3.5. *The quantum zonal spherical functions associated to the quantum Gelfand pair (A, B) are defined by*

$$\varphi_\alpha = \langle \pi_\alpha(v_\alpha), F_{V_\alpha}^{\frac{1}{2}} v_\alpha \rangle_\alpha \in A, \qquad \forall\, \alpha \in \Sigma_B,$$

where $0 \neq v_\alpha \in V_\alpha^B$ and $F_{V_\alpha}^{\frac{1}{2}}$ is the square root of the positive definite linear operator F_{V_α} on V_α.

Note that the quantum zonal spherical function φ_α ($\alpha \in \Sigma_B$) is uniquely determined up to nonzero scalar multiples. The square root of the positive definite operator F_{V_α} naturally appears in the definition of quantum zonal spherical functions on quantum homogeneous spaces using Noumi's [19] two-sided coideal approach. It is this approach, and its generalization in terms of coideal algebras by Letzter (see [17] and references therein), which has led to the identification of Macdonald polynomials as radial parts of quantum zonal spherical functions, see also Section 8 for the quantum complex Grassmannian.

It now follows from the preceding discussions that we have the following expression for the quantum dimension of V_α in terms of the associated quantum zonal spherical function φ_α and the associated B-invariant vector $v_\alpha \in V_\alpha^B$.

Proposition 3.6. *Let (A, B) be a quantum Gelfand pair. For $\alpha \in \Sigma_B$ we have*

$$\mathrm{Dim}_A(V_\alpha) = \frac{\|v_\alpha\|_\alpha^4}{\|\varphi_\alpha\|_h^2}.$$

Proof. This follows directly using the definition of φ_α and the generalized Schur orthogonality relations (Proposition 3.3b). $\qquad\square$

4. Grassmannians over local fields

In this section we specialize the general theory of the previous section to the classical examples arising from Grassmannians over local fields.

4.1. **The Grassmann representation over local fields.** Consider the commutative CQG algebra $A = \mathbb{C}(\mathcal{K})$ of representative functions on a compact topological group $\mathcal{K}$. The Hopf $*$-algebra structures are given by

$$\Delta(f)(g, g') = f(gg'), \quad \epsilon(f) = f(e), \quad S(f)(g) = f(g^{-1}), \quad f^*(g) = \overline{f(g)},$$

where $g, g' \in \mathcal{K}$, $f \in \mathbb{C}(\mathcal{K})$, $e \in \mathcal{K}$ is the unit element of $\mathcal{K}$, and where we have used the natural identification $\mathbb{C}(\mathcal{K}) \otimes \mathbb{C}(\mathcal{K}) \simeq \mathbb{C}(\mathcal{K} \times \mathcal{K})$ by the multiplication map. The positive definite normalized Haar functional h is

$$h(f) = \int_\mathcal{K} f(g) dg, \qquad f \in \mathbb{C}(\mathcal{K}),$$

where dg is the normalized Haar measure on $\mathcal{K}$. Observe that $\mathbb{C}(\mathcal{K})_h = L^2(\mathcal{K}, dg)$ and $\Sigma \simeq \widehat{\mathcal{K}}$, the unitary dual of the compact group $\mathcal{K}$, since there is a natural one-to-one correspondence between finite-dimensional right $\mathbb{C}(\mathcal{K})$-comodules and finite dimensional $\mathcal{K}$-representations preserving notions as irreducibility, equivalence and unitarity. Concretely, for a given right $\mathbb{C}(\mathcal{K})$-comodule $\pi : V \to V \otimes \mathbb{C}(\mathcal{K})$ we define a left $\mathcal{K}$-action on V by

$$gv = (\mathrm{Id}_V \otimes \mathrm{ev}_g)\pi(v), \qquad g \in \mathcal{K}, \; v \in V,$$

where $\mathrm{ev}_g : \mathbb{C}(\mathcal{K}) \to \mathbb{C}$ is the evaluation map $\mathrm{ev}_g(f) = f(g)$. Under this correspondence, the dual representation of a finite dimensional $\mathcal{K}$-representation (π, V) becomes

$$(\pi^*(g)f)(v) = f(\pi(g^{-1})v) \qquad (f \in V^*, v \in V, g \in \mathcal{K}).$$

Clearly the isomorphism $\pi^{**} \simeq \pi$ for a finite dimensional $\mathcal{K}$-representation (π, V) is realized by $F = \mathrm{Id}$, hence the quantum dimension $\mathrm{Dim}_{\mathbb{C}(\mathcal{K})}(V)$ is the complex dimension $\mathrm{Dim}_\mathbb{C}(V)$ of the representation space V. We now define one of our main object of study.

Definition 4.1. *Let $n, d \in \mathbb{N}$ with $n \leq d/2$. The Grassmannian $\mathcal{G}(n, d; \mathbb{F})$ over the local field $\mathbb{F}$ is the set of n-dimensional $\mathbb{F}$-subspaces in $\mathbb{F}^d$.*

Consider the maximal compact subgroup $\mathcal{K}_\mathbb{F}$ of $\mathrm{GL}_d(\mathbb{F})$, see (1.1). Note that $\mathcal{K}_\mathbb{F}$ acts transitively on $\mathcal{G}(n, d; \mathbb{F})$, e.g. by the Iwasawa decomposition in $\mathrm{GL}_d(\mathbb{F})$. The subgroup of $\mathcal{K}_\mathbb{F}$ stabilizing $\{0\}^{d-n} \times \mathbb{F}^n \in \mathcal{G}(n, d; \mathbb{F})$ is denoted by $\mathcal{M}_\mathbb{F}$, so that

$$\mathcal{G}(n, d; \mathbb{F}) \simeq \mathcal{K}_\mathbb{F}/\mathcal{M}_\mathbb{F}$$

as transitive $\mathcal{K}_\mathbb{F}$-space. The pair $(\mathcal{K}_\mathbb{F}, \mathcal{M}_\mathbb{F})$ is a Gelfand pair in the usual sense of topological groups. This is well known for the Archimedean fields $\mathbb{R}$ and $\mathbb{C}$, see [12] and [5] and the references therein. For non-Archimedean fields we refer to [2]. Consequently, $(A, B) = \big(\mathbb{C}(\mathcal{K}_\mathbb{F}), \mathbb{C}(\mathcal{M}_\mathbb{F})\big)$ is a quantum Gelfand pair in the sense of CQG algebras, where the surjective Hopf $*$-homomorphism $p : \mathbb{C}(\mathcal{K}_\mathbb{F}) \to \mathbb{C}(\mathcal{M}_\mathbb{F})$ is the canonical restriction map $p(f) = f|_{\mathcal{M}_\mathbb{F}}$. The parametrizing set $\Sigma_{\mathbb{C}(\mathcal{M}_\mathbb{F})}$ of the finite dimensional, irreducible $\mathbb{C}(\mathcal{M}_\mathbb{F})$-spherical right $\mathbb{C}(\mathcal{K}_\mathbb{F})$-comodules is in bijection with $(\mathcal{K}_\mathbb{F}/\mathcal{M}_\mathbb{F})\hat{\,}$, the unitary spherical dual of $\mathcal{K}_\mathbb{F}$ with respect to the subgroup $\mathcal{M}_\mathbb{F}$. The corresponding right $\mathbb{C}(\mathcal{K}_\mathbb{F})$-comodule $\mathbb{C}(\mathcal{K}_\mathbb{F})_{\mathbb{C}(\mathcal{M}_\mathbb{F})}$, viewed as $\mathcal{K}_\mathbb{F}$-representation, now identifies with the *Grassmann representation*

$$\mathbb{C}\big(\mathcal{G}(n, d; \mathbb{F})\big) := \{f \in \mathbb{C}(\mathcal{K}_\mathbb{F}) \mid f(gh) = f(g), \ \forall\, g \in \mathcal{K}_\mathbb{F}, \forall\, h \in \mathcal{M}_\mathbb{F}\},$$

considered as left $\mathcal{K}_\mathbb{F}$-module by the regular action

$$\big(gf\big)(g') = f(g^{-1}g'), \qquad f \in \mathbb{C}\big(\mathcal{G}(n, d; \mathbb{F})\big), \ g, g' \in \mathcal{K}_\mathbb{F}.$$

In this setting we take $(\pi_\alpha^\mathbb{F}, V_\alpha^\mathbb{F})$ $(\alpha \in (\mathcal{K}_\mathbb{F}/\mathcal{M}_\mathbb{F})\hat{\,})$ to be a complete set of representatives of the irreducible $\mathcal{M}_\mathbb{F}$-spherical $\mathcal{K}_\mathbb{F}$-representations. Then

$$(4.1) \qquad \mathbb{C}\big(\mathcal{G}(n, d; \mathbb{F})\big) \simeq \bigoplus_{\alpha \in (\mathcal{K}_\mathbb{F}/\mathcal{M}_\mathbb{F})\hat{\,}} V_\alpha^\mathbb{F}$$

is the multiplicity-free decomposition of $\mathbb{C}\big(\mathcal{G}(n, d; \mathbb{F})\big)$ in irreducible $\mathcal{K}_\mathbb{F}$-representations, cf. (3.2). For $\alpha \in (\mathcal{K}_\mathbb{F}/\mathcal{M}_\mathbb{F})\hat{\,}$ we choose a scalar product $\langle \cdot, \cdot \rangle_\alpha$ on $V_\alpha^\mathbb{F}$ turning $\pi_\alpha^\mathbb{F}$ into a unitary $\mathcal{K}_\mathbb{F}$-representation, and we choose a nonzero $\mathcal{M}_\mathbb{F}$-fixed vector $v_\alpha^\mathbb{F} \in V_\alpha^\mathbb{F}$. Using again the identification of finite dimensional right $\mathbb{C}(\mathcal{K}_\mathbb{F})$-comodules with finite dimensional $\mathcal{K}_\mathbb{F}$-representations, the quantum zonal spherical functions associated to the quantum Gelfand pair $(\mathbb{C}(\mathcal{K}_\mathbb{F}), \mathbb{C}(\mathcal{M}_\mathbb{F}))$ are the usual zonal spherical functions

$$\varphi_\alpha^\mathbb{F}(g) = \langle \pi_\alpha^\mathbb{F}(g) v_\alpha^\mathbb{F}, v_\alpha^\mathbb{F} \rangle_\alpha, \qquad g \in \mathcal{K}_\mathbb{F}, \ \alpha \in (\mathcal{K}_\mathbb{F}/\mathcal{M}_\mathbb{F})\hat{\,}$$

associated to the Gelfand pair $(\mathcal{K}_\mathbb{F}, \mathcal{M}_\mathbb{F})$. The dimension formula (see Proposition 3.6) now becomes

$$(4.2) \qquad \mathrm{Dim}_\mathbb{C}\big(V_\alpha^\mathbb{F}\big) = \frac{\big(\varphi_\alpha^\mathbb{F}(e)\big)^2}{\int_{\mathcal{K}_\mathbb{F}} |\varphi_\alpha^\mathbb{F}(g)|^2 dg}, \qquad \alpha \in (\mathcal{K}_\mathbb{F}/\mathcal{M}_\mathbb{F})\hat{\,},$$

where e is the unit element of $\mathcal{K}_\mathbb{F}$.

4.2. **Irreducible constituents and dimension formulas: the Archimedean case.** The harmonic analysis corresponding to the complex and real Grassmannian is classical by now. It relates to special cases of harmonic analysis on compact symmetric spaces. To emphasize the similarities with the non-Archimedean case and the quantum case, we now shortly describe the relevant results for the complex Grassmannian $\mathcal{G}(n, d; \mathbb{C})$, following closely the presentation in [5, Section 2]. For the real case, we refer to [12] and especially to [30] (the presentation in [30] is very close to the (quantum) complex case as discussed in this paper).

To avoid confusion with standard notations for the complexification of a compact Lie group, we write

$$(4.3) \qquad (\mathcal{K}, \mathcal{M}) = (\mathrm{U}(d), \mathrm{U}(d-n) \times \mathrm{U}(n))$$

for the Gelfand pair $(\mathcal{K}_{\mathbb{F}}, \mathcal{M}_{\mathbb{F}})$ when $\mathbb{F} = \mathbb{C}$. The irreducible $\mathcal{K}$-representations are parametrized by the cone

$$(4.4) \qquad P_d^+ = \{\mu \in \mathbb{Z}^d \mid \mu_1 \geq \mu_2 \geq \cdots \geq \mu_d\} \subset \mathbb{Z}^d,$$

by associating to $\mu \in P_d^+$ the $\mathcal{K}$-representation $L_\mu^{\mathbb{C}}$ of highest weight $\mu \in P_d^+$. The Weyl character formula states that the restriction of the character of $L_\mu^{\mathbb{C}}$ ($\mu \in P_d^+$) to the compact d-torus of diagonal matrices $\mathrm{diag}(u_1, \ldots, u_d)$ in $\mathrm{U}(d)$ is the Schur function

$$s_\mu(u_1, \ldots, u_d) = \Delta(u)^{-1} \sum_{w \in S_d} (-1)^{l(w)} u^{w(\mu + \rho_d)} \quad (\mu \in P_d^+),$$

where $\Delta(u) = \prod_{1 \leq i < j \leq d}(u_i - u_j)$ is the Vandermonde determinant, $l(w)$ is the length of $w \in S_n$, and $\rho_d = (d-1, d-2, \ldots, 0) \in P_d^+$ is the staircase partition. The dimension of $L_\mu^{\mathbb{C}}$ being the evaluation of its character at the unit element $e \in \mathcal{K}$ thus yields

$$(4.5) \qquad \mathrm{Dim}_{\mathbb{C}}(L_\mu^q) = s_\mu(1), \qquad \mu \in P_d^+.$$

The resulting, famous, Weyl dimension formula is

$$(4.6) \qquad \mathrm{Dim}_{\mathbb{C}}(L_\mu^{\mathbb{C}}) = \prod_{1 \leq i < j \leq d} \left(\frac{\mu_i - \mu_j + \delta_i - \delta_j}{\delta_i - \delta_j} \right), \qquad \mu \in P_d^+,$$

where

$$(4.7) \qquad \delta = \frac{1}{2}\big(d-1, d-3, \ldots, 3-d, 1-d\big)$$

is the half sum of positive roots in the standard realization of the A_{d-1}-type root system in $\mathbb{R}^d$. We note that other explicit dimension formulas are known, mostly with a combinatorial flavour. For instance, for $\mu \in P_d^+$ with $\mu_d \geq 0$, $\mathrm{Dim}_{\mathbb{C}}(L_\mu^{\mathbb{C}})$ is the number of standard Young tableaux of shape μ. Furthermore, $\mathrm{Dim}_{\mathbb{C}}(L_\mu^{\mathbb{C}})$ can be explicitly expressed in terms of products of hook-lengths of the Young diagram of shape μ (see [18]).

We state now a different type of dimension formula for the subclass of spherical irreducible representations associated to the complex Grassmannian $\mathcal{G}(n, d; \mathbb{C}) \simeq \mathcal{K}/\mathcal{M}$. This dimension formula is based on the expression (4.2) of the dimension in terms of the corresponding zonal spherical functions. The spherical unitary dual $(\mathcal{K}_{\mathbb{F}}/\mathcal{M}_{\mathbb{F}})^{\hat{}}$ naturally identifies with the set Λ_n of partitions of length at most n via the embedding

$$(4.8) \qquad \lambda^\natural = (\lambda_1, \lambda_2, \ldots, \lambda_n, \underbrace{0, \ldots, 0}_{d-2n}, -\lambda_n, \ldots, -\lambda_2, -\lambda_1)$$

of Λ_n in P_d^+. Consequently the irreducibles

$$V_\lambda^{\mathbb{C}} := L_{\lambda^\natural}^{\mathbb{C}}, \qquad (\lambda \in \Lambda_n)$$

form a complete set of representatives of the irreducible $\mathcal{M}_{\mathbb{F}}$-spherical $\mathcal{K}_{\mathbb{F}}$-representations, and the Grassmann representation $\mathbb{C}\big(\mathcal{G}(n,d;\mathbb{C})\big)$ has a multiplicity-free decomposition

$$\mathbb{C}\big(\mathcal{G}(n,d;\mathbb{C})\big) \simeq \bigoplus_{\lambda \in \Lambda_n} V_\lambda^{\mathbb{C}}$$

as $\mathcal{K}_{\mathbb{F}}$-module. We write $\varphi_\lambda^{\mathbb{C}}$ for the zonal spherical function corresponding to the irreducible representation $V_\lambda^{\mathbb{C}}$ ($\lambda \in \Lambda_n$).

Proposition 4.2. *For $\lambda \in \Lambda_n$ we have*

$$Dim_{\mathbb{C}}\big(V_\lambda^{\mathbb{C}}\big) = \prod_{i=1}^{n} \left(\frac{d - 2n + 1 + 2(\lambda_i + \rho_i)}{d - 2n + 1 + 2\rho_i} \right) \prod_{j=1}^{d-2n} \left(\frac{j + \lambda_i + \rho_i}{j + \rho_i} \right)^2$$

$$\times \prod_{1 \le j < k \le n} \left(\frac{d - 2n + 1 + \lambda_j + \lambda_k + \rho_j + \rho_k}{d - 2n + 1 + \rho_j + \rho_k} \right)^2 \left(\frac{\lambda_j - \lambda_k + \rho_j - \rho_k}{\rho_j - \rho_k} \right)^2,$$

where $\rho = (n-1, n-2, \ldots, 1, 0) \in \Lambda_n$ is the staircase partition.

The natural proof of the proposition uses the identification of the radial parts of the zonal spherical functions $\varphi_\lambda^{\mathbb{C}}$ ($\lambda \in \Lambda_n$) as BC_n-type Heckman-Opdam polynomials with appropriate multiplicity parameters (see e.g. [12] and [5, Section 2]). The dimension formula (4.2) then yields an expression in terms of evaluations and quadratic norms of Heckman-Opdam polynomials, which both have been explicitly evaluated.

Proposition 4.2 follows also as a direct consequence of the formula

(4.9) $$\mathrm{Dim}_{\mathbb{C}}\big(V_\lambda^{\mathbb{C}}\big) = D_{q^2}\big(\lambda; q^{2(d-2n+1)}, q^2; q^2\big)|_{q=1}, \qquad (\lambda \in \Lambda_n),$$

expressing the dimensions in terms of the classical ($q = 1$) degeneration of the generalized dimension formula $D_q(\lambda)$ (see (2.4)). Formula (4.9) is a direct consequence of the identification of multivariable little q-Jacobi polynomials with quantum zonal spherical functions on the quantum complex Grassmannian, see Subsection 5.3 for details. We note also that the classical ($q = 1$) degeneration of the multivariable little q-Jacobi polynomials have been studied in detail in [29], also in the context of harmonic analysis on (quantum) complex Grassmannians (see [5]).

Remark 4.3. Proposition 4.2 can easily be reconfirmed by specializing the Weyl dimension formula (4.6) to $\mu = \lambda^{\natural}$ ($\lambda \in \Lambda_n$).

For the real Grassmannian, $(\mathcal{K}_{\mathbb{R}}/\mathcal{M}_{\mathbb{R}})^{\hat{}}$ also naturally identifies with Λ_n, see [12], [23] and [30]. The harmonic analytic results from [12] lead to similar explicit dimension formulas for the associated irreducible $\mathcal{M}_{\mathbb{R}}$-spherical $\mathcal{K}_{\mathbb{R}}$-representations $V_\lambda^{\mathbb{R}}$ ($\lambda \in \Lambda_n$). In particular the dimensions can also be expressed as classical ($q = 1$) degenerations of generalized dimensions

(4.10) $$\mathrm{Dim}_{\mathbb{C}}\big(V_\lambda^{\mathbb{R}}\big) = D_{q^2}\big(\lambda; q^{d-2n+1}, q; q\big)|_{q=1}, \qquad (\lambda \in \Lambda_n),$$

which follow as in the complex case from the results in [12] and [29] (see also [23, §2.1.1]).

4.3. **Irreducible constituents and dimension formulas: the non-Archimedean case.** In this subsection we fix a non-Archimedean local field $\mathbb{F}$. Let $\mathcal{O}$ denote the ring of integers and $\mathfrak{p}$ the maximal ideal. The Grassmann representation $\mathbb{C}\big(\mathcal{G}(n,d;\mathbb{F})\big)$ was studied in [11] and later in [1, 2]. Therein it is shown that the irreducible $\mathcal{M}_{\mathbb{F}}$-spherical $\mathcal{K}_{\mathbb{F}}$-representations are also parameterized by Λ_n. We now proceed to recall the explicit construction of the irreducibles. The path to the construction of these representations is completely different compared to the Archimedean case. First of all, there is no (highest) weight theory, in particular there is no concrete description of the full unitary dual $\hat{\mathcal{K}}_{\mathbb{F}}$ as in the Archimedean case. We thus have to resort to a direct construction of the spherical unitary dual $(\mathcal{K}_{\mathbb{F}}/\mathcal{M}_{\mathbb{F}})^{\hat{}}$ and the concrete construction of the corresponding representations, which has a combinatorial flavor due to the profinite nature of $\mathcal{K}_{\mathbb{F}}$,

$$\mathcal{K}_{\mathbb{F}} \simeq \varprojlim \mathrm{GL}_d(\mathcal{O}/\mathfrak{p}^k).$$

The latter identification is realized by the canonical epimorphisms $\pi_k : \mathcal{K}_{\mathbb{F}} \to \mathrm{GL}_d(\mathcal{O}/\mathfrak{p}^k)$ defined by reduction modulo $\mathfrak{p}^k$ ($k \in \mathbb{Z}_{>0}$). Denoting I_k for the kernel of π_k, one has (cf. [2])

$$\mathbb{C}\big(\mathcal{G}(n,d;\mathbb{F})\big) \simeq \varinjlim \mathbb{C}\big(\mathcal{G}(n,d;\mathbb{F})\big)^{I_k}$$

as $\mathcal{K}_{\mathbb{F}}$-representations, where $\mathbb{C}\big(\mathcal{G}(n,d;\mathbb{F})\big)^{I_k}$ stands for the $\mathcal{K}_{\mathbb{F}}$-module consisting of the I_k-fixed vectors in the Grassmann representation. The structure of the Grassmann representation is completely determined by its so-called level $k-1$ components $\mathbb{C}\big(\mathcal{G}(n,d;\mathbb{F})\big)^{I_k}$ for $k \in \mathbb{Z}_{>0}$. We proceed now to recall the precise structure of these level components.

The set of partitions Λ_n parametrizes the isomorphism classes of the finite dimensional $\mathcal{O}$-modules of rank $\leq n$. Concretely, the partition $\lambda \in \Lambda_n$ corresponds to the isomorphism class of the finite $\mathcal{O}$-module $\oplus_{i=1}^n \mathcal{O}/\mathfrak{p}^{\lambda_i}$. The finite $\mathcal{O}$-modules in the isomorphism class labeled by $\lambda \in \Lambda_n$ are called $\mathcal{O}$-modules of type λ. Denote $\subseteq$ for the partial order on partitions defined by inclusion of Young diagrams. Fix a module of type $k^d = (k,\ldots,k)$ (d entries). For any type $\lambda \subseteq k^d$ let $\mathcal{G}(\lambda,k^d;\mathcal{O})$ be the Grassmannian of its submodules of type λ. Let $\mathbb{C}\big(\mathcal{G}(\lambda,k^d;\mathcal{O})\big)$ be the vector space of complex valued functions on $\mathcal{G}(\lambda,k^d;\mathcal{O})$. The canonical action of $\mathcal{K}_{\mathbb{F}}$ on $\mathcal{G}(\lambda,k^d;\mathcal{O})$ gives rise to a representation of $\mathcal{K}_{\mathbb{F}}$ on $\mathbb{C}\big(\mathcal{G}(\lambda,k^d;\mathcal{O})\big)$. The finite Grassmann representations relate to the Grassmann representation $\mathbb{C}\big(\mathcal{G}(n,d;\mathbb{F})\big)$ by the identification

$$(4.11) \qquad\qquad \mathbb{C}\big(\mathcal{G}(n,d;\mathbb{F})\big)^{I_k} \simeq \mathbb{C}\big(\mathcal{G}(k^n,k^d;\mathcal{O})\big)$$

as $\mathcal{K}_{\mathbb{F}}$-modules, see [2, Lemma 2.1].

The irreducible constituents of the finite Grassmann representation $\mathbb{C}\big(\mathcal{G}(k^n,k^d;\mathcal{O})\big)$ have been explicitly described in [2] in terms of the so-called cellular basis of the associated finite Hecke algebra $\mathrm{End}_{\mathcal{K}_{\mathbb{F}}}\big(\mathbb{C}\big(\mathcal{G}(k^n,k^d;\mathcal{O})\big)\big)$. It leads to a natural parametrization of the irreducible $\mathcal{M}_{\mathbb{F}}$-spherical $\mathcal{K}_{\mathbb{F}}$-representations by partitions Λ_n, as described in the following theorem, which is a direct consequence of [2, Theorem 1].

Denote Λ_n^k for partitions $\lambda \in \Lambda_n$ satisfying $\lambda \subseteq k^n$. Furthermore, for an irreducible $\mathcal{K}_{\mathbb{F}}$-module V we denote $\langle V : \mathbb{C}(\mathcal{G}(\mu, k^d; \mathcal{O})) \rangle$ for the multiplicity of V in the $\mathcal{K}_{\mathbb{F}}$-module $\mathbb{C}(\mathcal{G}(\mu, k^d; \mathcal{O}))$.

Theorem 4.4 ([2]). *For $\lambda \in \Lambda_n$ there exists a unique irreducible $\mathcal{M}_{\mathbb{F}}$-spherical $\mathcal{K}_{\mathbb{F}}$-representation $V_\lambda^{\mathbb{F}}$ satisfying, for all $k \in \mathbb{Z}_{>0}$ such that $\lambda \in \Lambda_n^k$,*

    (1) *$\langle V_\lambda^{\mathbb{F}} : \mathbb{C}(\mathcal{G}(\mu, k^d; \mathcal{O})) \rangle \geq 1$ for $\mu \in \Lambda_n^k$ satisfying $\lambda \subseteq \mu$,*

    (2) *$\langle V_\lambda^{\mathbb{F}} : \mathbb{C}(\mathcal{G}(\mu, k^d; \mathcal{O})) \rangle = 0$ for $\mu \in \Lambda_n^k$ satisfying $\lambda \nsubseteq \mu$.*

The $\mathcal{K}_{\mathbb{F}}$-representations $V_\lambda^{\mathbb{F}}$ ($\lambda \in \Lambda_n$) form a complete set of representatives of the irreducible $\mathcal{M}_{\mathbb{F}}$-spherical $\mathcal{K}_{\mathbb{F}}$-representations.

Theorem 4.4 gives the multiplicity free irreducible decomposition

$$\mathbb{C}(\mathcal{G}(n, d; \mathbb{F})) \simeq \bigoplus_{\lambda \in \Lambda_n} V_\lambda^{\mathbb{F}}$$

as $\mathcal{K}_{\mathbb{F}}$-modules, as well as the identifications

$$(\mathcal{K}_{\mathbb{F}}/\mathcal{M}_{\mathbb{F}})^{\hat{}} \simeq \Lambda_n \simeq \{\text{isomorphism types of } \mathcal{O}\text{-modules of rank} \leq n\}.$$

Furthermore, (4.11) and Theorem 4.4 imply

$$(4.12) \qquad \mathbb{C}(\mathcal{G}(k^n, k^d; \mathcal{O})) \simeq \bigoplus_{\lambda \in \Lambda_n^k} V_\lambda^{\mathbb{F}}$$

for $k \in \mathbb{Z}_{>0}$ as $\mathcal{K}_{\mathbb{F}}$-modules, cf. [2, Prop. 2.2]. The following is one of the main results in this paper.

Theorem 4.5. *Let $t = |\mathcal{O}/\mathfrak{p}|^{-1}$. We have*

$$(4.13) \qquad \mathrm{Dim}_{\mathbb{C}}(V_\lambda^{\mathbb{F}}) = t^{-(d-2n+1)|\lambda|-2(\rho,\lambda)} \begin{bmatrix} n \\ \partial\lambda' \end{bmatrix}_t \frac{(t^{d-\lambda_1'-\lambda_2'+2}; t)_{\lambda_1'+\lambda_2'}}{(t^{n-\lambda_1'+1}; t)_{\lambda_1'}} \frac{(1-t^{d-2\lambda_1'+1})}{(1-t^{d+1})}$$

for $\lambda \in \Lambda_n$.

Theorem 4.5 is a direct consequence of the formula

$$(4.14) \qquad \mathrm{Dim}_{\mathbb{C}}(V_\lambda^{\mathbb{F}}) = D_0(\lambda; t^{d-2n+1}, t; t), \qquad t = |\mathcal{O}/\mathfrak{p}|^{-1}$$

from [23], and the explicit evaluation of $D_0(\lambda; a, b; t)$ derived in Theorem 7.6. Formula (4.14) is obtained in [23] as a consequence of (4.2) and the identification of the zonal spherical function $\varphi_\lambda^{\mathbb{F}}$ with a suitable p-adic ($q = 0$) degeneration of the multivariable little q-Jacobi polynomial P_λ^L.

Corollary 4.6. *For $k \in \mathbb{Z}_{>0}$ we have*

$$\sum_{\lambda \subseteq k^n} t^{-(d-2n+1)|\lambda|-2(\rho,\lambda)} \begin{bmatrix} n \\ \partial\lambda' \end{bmatrix}_t \frac{(t^{d-\lambda_1'-\lambda_2'+2}; t)_{\lambda_1'+\lambda_2'}}{(t^{n-\lambda_1'+1}; t)_{\lambda_1'}} \frac{(1-t^{d-2\lambda_1'+1})}{(1-t^{d+1})} = t^{-n(d-n)(k-1)} \binom{d}{n}_{t^{-1}}.$$

Proof. Since the desired expression is rational in t, it suffices to prove the identity with the formal parameter t^{-1} specialized to the cardinality of the residue field of a non-Archimedean local field $\mathbb{F}$. In this case, the identity immediately follows from (4.12), (4.11), (4.13) and the easily verified identity (alternatively see e.g. [2, Claim 4.1(1)])

$$(4.15) \qquad |\mathcal{G}(k^n, k^d; \mathcal{O})| = t^{-n(d-n)(k-1)} \binom{d}{n}_{t^{-1}}, \qquad t = |\mathcal{O}/\mathfrak{p}|^{-1}.$$

$\square$

For $n = 1$, Corollary 4.6 reduces to the trivial geometric sum identity

$$1 + t^{1-d} \frac{(1 - t^{d-1})}{(1-t)} + \frac{(1 - t^{d-1})(1 - t^d)}{(1-t)} \sum_{m=2}^{k} t^{(1-d)m} = t^{(1-d)k} \frac{(1 - t^d)}{(1-t)}.$$

We end this subsection by discussing the dimension formula (4.13) in two cases that the dimensions have been computed before by representation theoretic methods. The dimension formula (4.13) yields for $r = 1, \ldots, n$,

$$(4.16) \qquad \mathrm{Dim}_{\mathbb{C}}\big(V_{\omega_r}^{\mathbb{F}}\big) = t^{r(r-d)} \frac{\big(t^{d-r+2}; t\big)_r (1 - t^{d-2r+1})}{\big(t; t\big)_r (1 - t^{d+1})} = \binom{d}{r}_{t^{-1}} - \binom{d}{r-1}_{t^{-1}}$$

where $t = |\mathcal{O}/\mathfrak{p}|^{-1}$. Alternatively, by (4.12)

$$\mathbb{C}\big(\mathcal{G}(\omega_r, 1^d; \mathcal{O})\big) \simeq \mathbb{C}\big(\mathcal{G}(\omega_{r-1}, 1^d; \mathcal{O})\big) \oplus V_{\omega_r}^{\mathbb{F}},$$

which implies (4.16) in view of (4.15). The latter representation theoretic derivation essentially is the argument from [6].

A similar analysis applies for the irreducible representations $V_k^{\mathbb{F}}$ ($k \in \mathbb{Z}_{\geq 2} \subset \Lambda_1$) arising in the representation $\mathbb{C}\big(\mathcal{G}(1, d; \mathbb{F})\big)$ associated to the projective space $\mathbb{P}^d(\mathbb{F}) = \mathcal{G}(1, d; \mathbb{F})$, see [9] and [23, §4]. In this case, (4.13) yields

$$\mathrm{Dim}_{\mathbb{C}}\big(V_k^{\mathbb{F}}\big) = t^{-(d-1)k} \frac{(1 - t^{d-1})(1 - t^d)}{(1-t)} = t^{-(d-1)k} \frac{(1 - t^d)}{(1-t)} - t^{-(d-1)(k-1)} \frac{(1 - t^d)}{(1-t)}$$

where $t = |\mathcal{O}/\mathfrak{p}|^{-1}$. This dimension formula can be reconfirmed using (4.15) and the decomposition

$$\mathbb{C}\big(\mathcal{G}(k^1, k^d; \mathcal{O})\big) \simeq \mathbb{C}\big(\mathcal{G}((k-1)^1, (k-1)^d; \mathcal{O})\big) \oplus V_k^{\mathbb{F}}$$

as $\mathcal{K}_{\mathbb{F}}$-modules, which is a direct consequence of (4.12).

5. The standard quantum complex Grassmannian

5.1. **The CQG algebra of quantum representative functions on $\mathcal{K}$.** Let $\mathcal{U}_q$ be the quantized universal enveloping algebra of $\mathfrak{gl}_n(\mathbb{C})$, where $0 < q < 1$. As a unital algebra it is generated by the analogs x_i, y_i ($i = 1, \ldots, d-1$) of the standard positive and negative simple root vectors of $\mathfrak{gl}_n(\mathbb{C})$ and by the Cartan type elements $K_j^{\pm 1}$ ($j = 1, \ldots, d$). For the defining relations, as well as for the explicit formulas turning $\mathcal{U}_q$ into a Hopf algebra, we refer to [22]. We view $\mathcal{U}_q$ as a Hopf $*$-algebra, with $*$-structure determined by

$$x_j^* = q^{-1} y_j K_j K_{j+1}^{-1}, \qquad y_j^* = q K_j^{-1} K_{j+1} x_j, \qquad (K_i^{\pm 1})^* = K_i^{\pm 1}.$$

A finite dimensional $\mathcal{U}_q$-representation decomposes in common eigenspaces for the action of the Cartan type generators K_j $(j = 1, \ldots, d)$. We call a finite dimensional $\mathcal{U}_q$-representation V of type one if

$$V = \bigoplus_{\nu \in \mathbb{Z}^d} V[\nu], \qquad V[\nu] = \{v \in V \mid K_j v = q^{\nu_j} v \quad (j = 1, \ldots, d)\}.$$

We say that a vector $0 \neq v \in V[\nu]$ has weight ν. We define $\mathbb{C}_q(\mathcal{K})$ to be the Hopf $*$-subalgebra of the Hopf dual of $\mathcal{U}_q$ spanned by the matrix coefficients of the finite dimensional $\mathcal{U}_q$-representations of type one. The Hopf $*$-algebra structure on $\mathbb{C}_q(\mathcal{K})$ is obtained by dualizing the Hopf $*$-algebra structure of $\mathcal{U}_q$, see [22]. In particular, $\mathbb{C}_q(\mathcal{K})$ inherits a $*$-structure from $\mathcal{U}_q$ by

$$f^*(X) = \overline{f(S(X)^*)}, \qquad f \in \mathbb{C}_q(\mathcal{K}), \ X \in \mathcal{U}_q,$$

where S is the antipode of $\mathcal{U}_q$. It is this particular choice of $*$-structure on $\mathbb{C}_q(\mathcal{K})$ that reflects the fact that we are considering quantized functions on the compact real form $\mathcal{K}$ of $\mathrm{GL}_d(\mathbb{C})$.

As in the classical case, which we discussed in Subsection 4.2, there is a one-to-one correspondence between finite dimensional (irreducible) right $\mathbb{C}_q(\mathcal{K})$-comodules and finite dimensional (irreducible) $\mathcal{U}_q$-representations of type one (see [21]). Concretely, for a finite dimensional right $\mathbb{C}_q(\mathcal{K})$-comodule $\pi : V \to V \otimes \mathbb{C}_q(\mathcal{K})$, we define the associated $\mathcal{U}_q$-action on V by

$$Xv = (\mathrm{Id} \otimes \mathrm{ev}_X)(\pi(v)), \qquad X \in \mathcal{U}_q, \ v \in V$$

where $\mathrm{ev}_X : \mathbb{C}_q(\mathcal{K}) \to \mathbb{C}$ is the evaluation map $\mathrm{ev}_X(f) = f(X)$. Under this identification, the $\mathcal{U}_q$-representation V^* dual to V is given by

$$(Xf)(v) = f(S(X)v), \qquad X \in \mathcal{U}_q, \ f \in V^*, \ v \in V.$$

The equivalence classes of the finite dimensional, irreducible $\mathcal{U}_q$-representations of type one are again naturally parameterized by the cone P_d^+ (4.4). The corresponding irreducible $\mathcal{U}_q$-representation L_μ^q $(\mu \in P_d^+)$ is characterized as the irreducible representation with highest weight μ. In particular, L_μ^q $(\mu \in P_d^+)$ has a (unique up to scalar multiples) weight vector $0 \neq u_\mu \in L_\mu^q[\mu]$ satisfying

$$x_i u_\mu = 0, \qquad i = 1, \ldots, d-1,$$

called a highest weight vector. Under the above mentioned identification of right $\mathbb{C}_q(\mathcal{K})$-comodules and left $\mathcal{U}_q$-modules, we have the Peter-Weyl decomposition

$$\tag{5.1} \mathbb{C}_q(\mathcal{K}) \simeq \bigoplus_{\mu \in P_d^+} L_\mu^{q\,*} \otimes L_\mu^q,$$

where $\phi \otimes v \in L_\mu^{q\,*} \otimes L_\mu^q$ is identified with the matrix coefficient $\phi(\cdot v) \in \mathbb{C}_q(\mathcal{K})$, which gives the multiplicity free irreducible decomposition as right $\mathbb{C}_q(\mathcal{K}) \otimes \mathbb{C}_q(\mathcal{K})$-comodules (see [21]). Consequently $\mathbb{C}_q(\mathcal{K})$ is a CQG algebra, with the normalized, positive definite Haar functional h on $\mathbb{C}_q(\mathcal{K})$ defined by requiring that it vanishes on the matrix coefficients

of the nontrivial irreducible representations L_μ^q ($0 \neq \mu \in P_d^+$). It follows from the Peter-Weyl decomposition that $\Sigma \simeq P_d^+$, as for the classical algebra of representative functions on $\mathcal{K}$. The quantum Schur orthogonality relations (Proposition 3.3) for the CQG algebra $A = \mathbb{C}_q(\mathcal{K})$ are well known, see e.g. [21, Section 3.2]. To make the link precise, note that the quantum dimension

$$\mathrm{Dim}_q(V) := \mathrm{Dim}_{\mathbb{C}_q(\mathcal{K})}(V)$$

of a finite dimensional irreducible $\mathcal{U}_q$-representation V of type one has the following explicit form.

Lemma 5.1. *Let (π, V) be a finite dimensional irreducible $\mathcal{U}_q$-representation of type one. Then*

$$\mathrm{Dim}_q(V) = \mathrm{Tr}_V(K^{2\delta} \cdot).$$

Proof. It is well known and easy to check that

$$S^2(X) = K^{2\delta} X K^{-2\delta}, \qquad \forall X \in \mathcal{U}_q.$$

Consequently, the isomorphism $\pi^{**} \simeq \pi$ as $\mathcal{U}_q$-modules is realized by the linear operator

$$F = \pi(K^{-2\delta}) = \pi\left(K_1^{1-d} K_2^{3-d} \cdots K_d^{d-1}\right).$$

Turning V into a $*$-unitary representation, the linear operator F becomes a positive definite linear operator on V satisfying $\mathrm{Tr}_V(F) = \mathrm{Tr}_V(F^{-1})$ (by an easy weight argument). The result now follows from the definition of the quantum dimension (Definition 3.2). $\qquad\square$

Remark 5.2. The quantized universal enveloping algebra $\mathcal{U}_q$ is a ribbon algebra. Lemma 5.1 implies that the quantum dimension $\mathrm{Dim}_q(V)$ of an irreducible, finite dimensional $\mathcal{U}_q$-module coincides with the natural (topologically motivated) notion of quantum dimension for finite dimensional modules over a ribbon algebra, see [13, Chapter XIV].

Since $\mathrm{Dim}_{\mathbb{C}}(L_\mu^q[\nu]) = \mathrm{Dim}_{\mathbb{C}}(L_\mu^{\mathbb{C}}[\nu])$ for all $\nu \in \mathbb{Z}^d$, one has

$$(5.2) \qquad \mathrm{Dim}_q(L_\mu^q) = \sum_{\nu \in \mathbb{Z}^d} \mathrm{Dim}_{\mathbb{C}}(L_\mu^{\mathbb{C}}[\nu]) q^{2\langle \delta, \nu \rangle} = s_\mu(q^{2\delta}), \qquad \mu \in P_d^+.$$

This implies the following q-analogue of the Weyl dimension formula,

$$(5.3) \qquad \mathrm{Dim}_q(L_\mu^q) = q^{-2\langle \delta, \mu \rangle} \prod_{1 \leq i < j \leq d} \left(\frac{1 - q^{2(\mu_i - \mu_j + \delta_i - \delta_j)}}{1 - q^{2(\delta_i - \delta_j)}} \right),$$

see e.g. [25, Lemma 2.5] for a simple proof based on the classical Weyl denominator formula. A combinatorial formula for the quantum dimension $\mathrm{Dim}_q(L_\mu^q)$ ($\mu \in P_d^+$) is provided by the q-analogue of the hook-length formula, see [18] and [21, Section 3.2] for more details.

5.2. **The quantum Grassmann representation.** In this subsection we discuss the standard quantum analog of the complex Grassmannian $\mathcal{G}(n, d; \mathbb{C}) \simeq \mathcal{K}/\mathcal{M}$ and its associated quantum Grassmann representation $\mathbb{C}_q(\mathcal{K}/\mathcal{M})$. The one-parameter family of quantum complex Grassmannians from [20] is discussed separately in Section 8.

We define a Gelfand pair $(\mathbb{C}_q(\mathcal{K}), \mathbb{C}_q(\mathcal{M}))$ of CQG algebras as follows. Write $\mathfrak{m}^{\mathbb{C}} \simeq \mathfrak{gl}_{d-n}(\mathbb{C}) \times \mathfrak{gl}_n(\mathbb{C})$ for the complexified Lie algebra of $\mathcal{M}$. We view $\mathfrak{m}^{\mathbb{C}}$ as Lie subalgebra of $\mathfrak{gl}_d(\mathbb{C})$ in the standard way and write

$$\mathcal{U}_q(\mathfrak{m}^{\mathbb{C}}) \simeq \mathcal{U}_q(\mathfrak{gl}_{d-n}(\mathbb{C})) \otimes \mathcal{U}_q(\mathfrak{gl}_n(\mathbb{C}))$$

for the Hopf $*$-subalgebra of $\mathcal{U}_q = \mathcal{U}_q(\mathfrak{gl}_d(\mathbb{C}))$ generated by $K_i^{\pm 1}$ ($i = 1, \ldots, d$) and x_j, y_j ($j \in \{1, \ldots, d-1\} \setminus \{d-n\}$). Let $\mathbb{C}_q(\mathcal{M})$ be the span of the matrix coefficients of the finite dimensional irreducible $\mathcal{U}_q(\mathfrak{m}^{\mathbb{C}})$-representations of type one. Then $(\mathbb{C}_q(\mathcal{M}), p)$ is a quantum subgroup of $\mathbb{C}_q(\mathcal{K})$, with p the surjective Hopf $*$-algebra morphism $p : \mathbb{C}_q(\mathcal{K}) \to \mathbb{C}_q(\mathcal{M})$ defined by $p(f) = f|_{\mathcal{U}_q(\mathfrak{m}^{\mathbb{C}})}$.

Suppose V is a finite dimensional right $\mathbb{C}_q(\mathcal{K})$-comodule. Using the identification of right $\mathbb{C}_q(\mathcal{K})$-comodules and left $\mathcal{U}_q$-modules, the subspace of $\mathbb{C}_q(\mathcal{M})$-invariant elements in V coincides with the subspace of $\mathcal{U}_q(\mathfrak{m}^{\mathbb{C}})$-invariant elements in V,

$$V^{\mathbb{C}_q(\mathcal{M})} = \{v \in V \mid Xv = \epsilon(X)v \quad \forall X \in \mathcal{U}_q(\mathfrak{m}^{\mathbb{C}})\}.$$

In translating the general CQG algebra theory from right $\mathbb{C}_q(\mathcal{K})$-modules to left $\mathcal{U}_q$-representations, we shall refer to $\mathbb{C}_q(\mathcal{K})$-spherical representations as $\mathcal{U}_q(\mathfrak{m}^{\mathbb{C}})$-spherical representations. The branching rules for finite dimensional $\mathcal{U}_q$-representations of type one, when viewed as representations of $\mathcal{U}_q(\mathfrak{m}^{\mathbb{C}}) \subset \mathcal{U}_q$, are the same as for the classical Gelfand pair $(\mathcal{K}, \mathcal{M})$. Consequently $(\mathbb{C}_q(\mathcal{K}), \mathbb{C}_q(\mathcal{M}))$ is a quantum Gelfand pair, with associated unitary spherical dual $\Sigma_{\mathbb{C}_q(\mathcal{M})} \simeq (\mathcal{K}/\mathcal{M})\hat{} \simeq \Lambda_n$ (compare with the classical complex Grassmannian of Subsection refcG, e.g. (4.8)). In particular, the $\mathcal{U}_q$-representations

$$V_\lambda^q := L_{\lambda^\natural}^q, \quad (\lambda \in \Lambda_n)$$

form a complete set of representatives of the finite dimensional, irreducible $\mathcal{U}_q(\mathfrak{m}^{\mathbb{C}})$-spherical $\mathcal{U}_q$-representations. The right $\mathbb{C}_q(\mathcal{K})$-comodule $\mathbb{C}_q(\mathcal{K})_{\mathbb{C}_q(\mathcal{M})}$ now identifies with the *quantum Grassmann representation*

$$\mathbb{C}_q(\mathcal{K}/\mathcal{M}) := \{f \in \mathbb{C}_q(\mathcal{K}) \mid f(XY) = f(X) \quad \forall X \in \mathcal{U}_q, \forall Y \in \mathcal{U}_q(\mathfrak{m}^{\mathbb{C}})\},$$

viewed as left $\mathcal{U}_q$-module by the regular $\mathcal{U}_q$-action

$$(Xf)(X') = f(S(X)X'), \qquad f \in \mathbb{C}_q(\mathcal{K}/\mathcal{M}), \ X, X' \in \mathcal{U}_q.$$

It has the multiplicity free decomposition

$$\mathbb{C}_q(\mathcal{K}/\mathcal{M}) \simeq \bigoplus_{\lambda \in \Lambda_n} V_\lambda^q$$

in irreducible $\mathcal{U}_q$-representations.

For $\lambda \in \Lambda_n$ we fix a nonzero $\mathcal{U}_q(\mathfrak{m}^{\mathbb{C}})$-invariant vector $v_\lambda^q \in V_\lambda^q$, and we choose a scalar product $\langle \cdot, \cdot \rangle_\lambda$ on V_λ^q which turns V_λ^q into a $*$-unitary $\mathcal{U}_q$-representation. The quantum zonal spherical functions $\varphi_\lambda^q \in \mathbb{C}_q(\mathcal{K}/\mathcal{M})$ ($\lambda \in \Lambda_n$) are given by

$$\varphi_\lambda^q(X) = \langle X v_\lambda^q, K^{-\delta} v_\lambda^q \rangle_\lambda, \qquad X \in \mathcal{U}_q,$$

where $K^{\pm\delta}$ is the linear endomorphism of V_λ^q satisfying

$$K^{\pm\delta}|_{V_\lambda^q[\nu]} = q^{\pm\langle\delta,\nu\rangle} \mathrm{Id}_{V_\lambda^q[\nu]}, \qquad \forall \nu \in \mathbb{Z}^d$$

(which is the square root of the positive definite linear operator on V_λ^q defined by the action of $K^{\pm 2\delta}$ on V_λ^q). We define for $\psi \in \mathrm{End}_{\mathbb{C}}(V_\lambda^q)$,

$$\varphi_\lambda^q(\psi) := \langle \psi(v_\lambda^q), K^{-\delta} v_\lambda^q \rangle_\lambda,$$

in particular $\varphi_\lambda^q(K^\delta) = \|v_\lambda^q\|_\lambda^2$. The most convenient form of the quantum dimension formula (Proposition 3.6) then reads

$$(5.4) \qquad \mathrm{Dim}_q\big(V_\lambda^q\big) = \frac{\big(\varphi_\lambda^q(K^\delta)\big)^2}{\|\varphi_\lambda^q\|_h^2}, \qquad \lambda \in \Lambda_n.$$

5.3. **Quantum dimension formulas.** The quantum dimensions of the irreducible spherical $\mathcal{U}_q$-representations V_λ^q ($\lambda \in \Lambda_n$) have the following explicit closed expressions.

Proposition 5.3. *For $\lambda \in \Lambda_n$ we have*

$$(5.5) \qquad \begin{aligned} Dim_q\big(V_\lambda^q\big) &= q^{2(2n-d-1)|\lambda|-4(\rho,\lambda)} \prod_{i=1}^n \frac{\big(q^{2(1+\lambda_i+\rho_i)}; q^2\big)_{d-2n}^2}{\big(q^{2(1+\rho_i)}; q^2\big)_{d-2n}^2} \frac{(1-q^{2(d-2n+1+2(\lambda_i+\rho_i))})}{(1-q^{2(d-2n+1+2\rho_i)})} \\ &\times \prod_{1 \le j < k \le n} \frac{(1-q^{2(d-2n+1+\lambda_j+\lambda_k+\rho_j+\rho_k)})^2(1-q^{2(\lambda_j-\lambda_k+\rho_j-\rho_k)})^2}{(1-q^{2(d-2n+1+\rho_j+\rho_k)})^2(1-q^{2(\rho_j-\rho_k)})^2}. \end{aligned}$$

In Section 8 we derive Proposition 5.3 from a generalization of the quantum dimension formula (5.4) to the setup of the one-parameter family of quantum complex Grassmannians from [20] and [5]. By a suitable degeneration procedure we subsequently deduce the quantum dimension formulas

$$(5.6) \qquad \mathrm{Dim}_q\big(V_\lambda^q\big) = D_{q^2}\big(\lambda; q^{2(d-2n+1)}, q^2; q^2\big), \qquad \lambda \in \Lambda_n$$

in terms of evaluations and quadratic norms of multivariable little q-Jacobi polynomials. More directly, (5.6) is a direct consequence of the quantum dimension formula (5.4) if one uses the fact that the quantum zonal spherical functions φ_λ^q ($\lambda \in \Lambda_n$) identify with multivariable little q-Jacobi polynomials (see [5] and Remark 8.5).

Remark 5.4. Taking the limit $q \to 1$ in (5.5), we obtain the explicit dimension formulas for the irreducible $\mathcal{M}$-spherical $\mathcal{K}$-representations $V_\lambda^{\mathbb{C}}$ ($\lambda \in \Lambda_n$) as stated in Proposition 4.2.

We end this subsection by discussing the representation theoretic viewpoint on the quantum dimension formula (5.5) for $\lambda = \omega_r$, in which case it yields the explicit expression

$$(5.7) \qquad \mathrm{Dim}_q\big(V_{\omega_r}^q\big) = q^{2r(d-r)} \binom{d}{r}_{q^{-2}}^2 - q^{2(r-1)(d-r+1)} \binom{d}{r-1}_{q^{-2}}^2$$

for $r = 1, \ldots, n$ by a straightforward computation. The following representation theoretic derivation of (5.7) bears strong resemblance to the representation theoretic derivation of the dimension formula (4.16) in the classical non-Archimedean setup.

Let $\{\epsilon_j\}_{j=1}^d$ be the standard orthonormal basis of $\mathbb{R}^d$. Let $V = L_{\epsilon_1}^q$ be the d-dimensional vector-representation of $\mathcal{U}_q$. Its weight spaces are one-dimensional, with weights ϵ_j ($j = 1, \ldots, d$). The irreducible representation $L_{\epsilon_1 + \cdots + \epsilon_r}^q$ ($1 \leq r \leq n$) can be realized as the rth graded part $\Lambda_q^r(V)$ of the q-exterior algebra $\Lambda_q(V)$ of V (see [21]). The weight spaces of $\Lambda_q^r(V)$ are again one-dimensional, with weights given by $\sum_{j \in J} \epsilon_j$ for subsets $J \subseteq \{1, \ldots, d\}$ of cardinality r. Hence

$$\mathrm{Dim}_q\big(\Lambda_q^r(V)\big) = \mathrm{Tr}_{\Lambda_q^r(V)}(K^{2\delta}\cdot) = \sum_{\substack{J \subseteq \{1,\ldots,d\} \\ \#J = r}} \prod_{j \in J} q^{2(d-j)} = q^{(r-1)r} \frac{\big(q^{2(d-r+1)}; q^2\big)_r}{\big(q^2; q^2\big)_r},$$

where the third equality follows by an easy induction argument. The dual of $\Lambda_q^r(V)$ is irreducible of highest weight $-\epsilon_{d-r+1} \cdots - \epsilon_d$. In fact, by [21] we have

$$\Lambda_q^r(V)^* \simeq \mathbb{C}\mathrm{det}_q^{-1} \otimes \Lambda_q^{d-r}(V),$$

with $\mathbb{C}\mathrm{det}_q^{-1} \simeq L_{-\epsilon_1 \cdots -\epsilon_d}^q$ the one-dimensional representation realized by the inverse of the quantum determinant $\mathrm{det}_q \in \mathbb{C}_q(\mathcal{K})$, hence

$$\mathrm{Dim}_q\big(\Lambda_q^r(V)^*\big) = q^{(1+r-2d)r} \frac{\big(q^{2(r+1)}; q^2\big)_{d-r}}{\big(q^2; q^2\big)_{d-r}}.$$

We conclude that

$$\mathrm{Dim}_q\big(U_r\big) = q^{2r(d-r)} \binom{d}{r}_{q^{-2}}^2,$$

where U_r is the $\mathcal{U}_q$-module

$$U_r = \Lambda_q^r(V) \otimes \Lambda_q^r(V)^*.$$

By [5, (6.14)] we have

$$U_r \simeq U_{r-1} \oplus V_{\omega_r}^q, \qquad r = 1, \ldots, n$$

as $\mathcal{U}_q$-modules (with U_0 the trivial representation), which now immediately implies (5.7).

6. Evaluation formulas

In this section we derive evaluation formulas for multivariable little and big q-Jacobi polynomials by degenerating evaluation formulas for the Macdonald-Koornwinder polynomials [15]. Evaluation formulas for Macdonald-Koornwinder polynomials have been obtained in [3], [24] and [28]. The evaluation formulas for the multivariable little q-Jacobi polynomials

are used in the next section to obtain explicit expressions for the generalized dimensions D_q (see (2.4)) and their degenerations.

6.1. **Macdonald-Koornwinder polynomials.** Recall our notation $\mathbb{E} = \mathbb{C}(a, b, c, d, q, t)$ for the field of rational functions in six indeterminates a, b, c, d, q and t. Let $W = S_n \ltimes \{\pm 1\}^n$, acting on $\mathbb{Z}^n$ by permutations and sign-changes of the coordinates, and acting on the algebra $\mathbb{E}[z^{\pm 1}] = \mathbb{E}[z_1^{\pm 1}, \ldots, z_n^{\pm 1}]$ by permutations and inversions of the independent variables z_i. The cone $\Lambda_n \subset \mathbb{Z}^n$ is a fundamental domain for the W-action on $\mathbb{Z}^n$. The dominance order on Λ_n is the partial order $\leq$ defined by $\lambda \leq \mu$ if $\sum_{j=1}^i \lambda_j \leq \sum_{j=1}^i \mu_j$ for $1 \leq i \leq n$. An $\mathbb{E}$-basis of the algebra $\mathbb{E}[z^{\pm 1}]^W$ of W-invariant Laurent polynomials is given by the symmetric monomials $m_\lambda(z) := \sum_{\mu \in W\lambda} z^\mu$ ($\lambda \in \Lambda_n$).

The monic Macdonald-Koornwinder polynomial $P_\lambda(z) = P_\lambda(z; a, b, c, d; q, t) \in \mathbb{E}[z^{\pm 1}]^W$ of degree $\lambda \in \Lambda_n$ is of the form

$$(6.1) \qquad P_\lambda(z) = m_\lambda(z) + \sum_{\mu \in \Lambda_n : \mu < \lambda} c_{\lambda,\mu} m_\mu(z)$$

for certain coefficients $c_{\lambda,\mu} = c_{\lambda,\mu}(a, b, c, d; q, t) \in \mathbb{E}$. It can be characterized as solution of a particular second-order difference equation, or, for suitably specialized generic parameters a, b, c, d, q, t, in terms of suitable orthogonality properties, see Koornwinder [15]. The characterizing difference equation for the Macdonald-Koornwinder polynomial $P_\lambda(z)$ of degree λ is $DP_\lambda = E_\lambda P_\lambda$ where

$$D = \sum_{j=1}^n \big(\phi_j(z)(T_j - \mathrm{Id}) + \phi_j(z^{-1})(T_j^{-1} - \mathrm{Id})\big),$$

$$\phi_j(z) = \frac{(1 - az_j)(1 - bz_j)(1 - cz_j)(1 - dz_j)}{(1 - z_j^2)(1 - qz_j^2)} \prod_{l \neq j} \frac{(1 - tz_l z_j)(1 - tz_l^{-1} z_j)}{(1 - z_l z_j)(1 - z_l^{-1} z_j)}$$

and where the eigenvalue $E_\lambda \in \mathbb{E}$ is given by

$$E_\lambda = \sum_{j=1}^n \big(q^{-1} abcdt^{2n-j-1}(q^{\lambda_j} - 1) + t^{j-1}(q^{-\lambda_j} - 1)\big).$$

Note that $P_\lambda(z; a, b, c, d; q, t)$ is symmetric in the four parameters a, b, c, d. The evaluation formulas for Macdonald-Koornwinder polynomials are given by

$$(6.2) \qquad \begin{aligned} P_\lambda(at^\rho) = &\prod_{i=1}^n \frac{\big(abt^{n-i}, act^{n-i}, adt^{n-i}, q^{-1}abcdt^{n-i}; q\big)_{\lambda_i}}{\big(q^{-1}abcdt^{2(n-i)}; q\big)_{2\lambda_i} \big(at^{n-i}\big)^{\lambda_i}} \\ &\times \prod_{1 \leq j < k \leq n} \frac{\big(q^{-1}abcdt^{2n-j-k+1}; q\big)_{\lambda_j + \lambda_k} \big(t^{k-j+1}; q\big)_{\lambda_j - \lambda_k}}{\big(q^{-1}abcdt^{2n-j-k}; q\big)_{\lambda_j + \lambda_k} \big(t^{k-j}; q\big)_{\lambda_j - \lambda_k}} \end{aligned}$$

for $\lambda \in \Lambda_n$. The evaluation formulas (6.2) have been established by van Diejen [3] for a sub-family of Macdonald-Koornwinder polynomials. The general case follow from Sahi's [24] results using double affine Hecke algebras, see also [28, Remark 9.5].

In the rank one case ($n = 1$), the Macdonald-Koornwinder polynomials do not dependent on t and reduce to the monic Askey-Wilson polynomials

$$P_m(z) = \frac{(ab, ac, ad; q)_m}{a^m(q^{m-1}abcd; q)_m}\, {}_4\phi_3\left(\begin{matrix} q^{-m}, q^{m-1}abcd, az, az^{-1} \\ ab, ac, ad \end{matrix}; q, q\right) \in \mathbb{C}[z + z^{-1}], \qquad m \in \mathbb{Z}_+.$$

In particular

$$P_m(a) = \frac{(ab, ac, ad; q)_m}{a^m(q^{m-1}abcd; q)_m}, \qquad m \in \mathbb{Z}_+,$$

which is in accordance to (6.2).

We give here a purely algebraic formulation of the quadratic norm evaluations for the Macdonald-Koornwinder polynomials, which has the advantage that we do not need to specialize the parameters a, b, c, d, q and t. We define an $\mathbb{E}$-linear functional

$$h_K = h_K^{a,b,c,d;q,t} : \mathbb{E}[z^{\pm 1}]^W \to \mathbb{E}$$

by requiring

$$h_K(P_\lambda) = \begin{cases} 1, & \text{if } \lambda = 0, \\ 0, & \text{if } \lambda \neq 0. \end{cases}$$

An analytic definition of h_K can be given (for specific Zariski dense choice of parameters) as an integral over a deformed compact n-torus with explicit weight function, see e.g. [15]. We define now a sesqui-linear form $\langle \cdot, \cdot \rangle_K = \langle \cdot, \cdot \rangle_K^{a,b,c,d;q,t}$ on $\mathbb{E}[z^{\pm 1}]^W$ by

$$(6.3) \qquad \langle p_1, p_2 \rangle_K = h_K\big(p_1(z)\overline{p_2(z)}\big), \qquad p_1, p_2 \in \mathbb{E}[z^{\pm 1}]^W,$$

where complex conjugation on $\mathbb{C}$ is extended to an anti-linear algebra involution on $\mathbb{E}[z^{\pm 1}]$ by requiring the six parameters a, b, c, d, q, t to be formally real, e.g. $\overline{a} = a$, and by requiring the variables z_j to be formally purely imaginary, $\overline{z_j} = z_j^{-1}$. The orthogonality relations for the Macdonald-Koornwinder polynomials are now given by

$$(6.4) \qquad \langle P_\lambda, P_\mu \rangle_K = N_K(\lambda)\delta_{\lambda,\mu}, \qquad \forall \lambda, \mu \in \Lambda_n,$$

with the quadratic norms $N_K(\lambda) = N_K(\lambda; a, b, c, d; q, t)$ given by $N_K(\lambda) = N_K^+(\lambda)N_K^-(\lambda)$, where $N_K^\pm(\lambda) = N_K^\pm(\lambda; a, b, c, d; q, t) \in \mathbb{E}$ are

$$N_K^+(\lambda) = \prod_{i=1}^n \frac{(abt^{n-i}, act^{n-i}, adt^{n-i}, q^{-1}abcdt^{n-i}; q)_{\lambda_i}}{(q^{-1}abcdt^{2(n-i)}; q)_{2\lambda_i}}$$

$$\times \prod_{1 \le j < k \le n} \frac{(q^{-1}abcdt^{2n-j-k+1}; q)_{\lambda_j+\lambda_k}(t^{k-j+1}; q)_{\lambda_j-\lambda_k}}{(q^{-1}abcdt^{2n-j-k}; q)_{\lambda_j+\lambda_k}(t^{k-j}; q)_{\lambda_j-\lambda_k}},$$

$$N_K^-(\lambda) = \prod_{i=1}^n \frac{(qt^{n-i}, bct^{n-i}, bdt^{n-i}, cdt^{n-i}; q)_{\lambda_i}}{(abcdt^{2(n-i)}; q)_{2\lambda_i}}$$

$$\times \prod_{1 \le j < k \le n} \frac{(abcdt^{2n-j-k-1}; q)_{\lambda_j+\lambda_k}(qt^{k-j-1}; q)_{\lambda_j-\lambda_k}}{(abcdt^{2n-j-k}; q)_{\lambda_j+\lambda_k}(qt^{k-j}; q)_{\lambda_j-\lambda_k}}.$$

The quadratic norm formulas have been established by van Diejen [3] for a sub-family of Macdonald-Koornwinder polynomials. The general case has been derived in [24], see also [28].

6.2. **Multivariable little q-Jacobi polynomials.** We now return to the multivariable little q-Jacobi polynomials $P_\lambda^L(z) = P_\lambda^L(z; a, b; q, t)$ from Section 2. We first shortly discuss evaluation formulas for one-variable little q-Jacobi polynomials ($n = 1$), in which case the evaluation formulas can be derived directly from the explicit expressions of the little q-Jacobi polynomials as basic hypergeometric series. In fact, the monic one-variable little q-Jacobi polynomial is independent of t and is given explicitly as

$$
(6.5) \quad
\begin{aligned}
P_m^L(z) &= \frac{(qb; q)_m}{(q^{m+1}ab; q)_m (qb)^m} {}_3\phi_2 \left(\begin{matrix} q^{-m}, q^{m+1}ab, qbz \\ qb, 0 \end{matrix}; q, q \right) \\
&= \frac{(qa; q)_m}{(q^{m+1}ab; q)_m} (-1)^m q^{\frac{1}{2}m(m-1)} {}_2\phi_1 \left(\begin{matrix} q^{-m}, q^{m+1}ab \\ qa \end{matrix}; q, qz \right)
\end{aligned}
$$

for $m \in \mathbb{Z}_+$, see e.g. [14]. Here the second equality follows from [8, (III.7)]. From the second equality in (6.5) we obtain

$$
(6.6) \quad P_m^L(0) = \frac{(qa, qab; q)_m}{(qab; q)_{2m}} (-1)^m q^{\frac{1}{2}m(m-1)}.
$$

From the first equality in (6.5) and the q-Vandermonde sum

$$
(6.7) \quad {}_2\phi_1 \left(\begin{matrix} q^{-m}, a \\ c \end{matrix}; q, q \right) = \frac{(c/a; q)_m}{(c; q)_m} a^m
$$

for $m \in \mathbb{Z}_+$ (see [8, (II.6)]), we obtain

$$
(6.8) \quad P_m^L(1) = \frac{(qb, qab; q)_m}{(qab; q)_{2m}} (q^m a)^m.
$$

Finally, from the first equality in (6.5) we obtain

$$
(6.9) \quad P_m^L(q^{-1}b^{-1}) = \frac{(qb, qab; q)_m}{(qab; q)_{2m} (qb)^m}.
$$

In the following theorem we give the multivariable analogues of the evaluation formulas (6.6), (6.8) and (6.9). Define $\Delta_\lambda = \Delta_\lambda(a, b; q, t)$ by

$$
(6.10) \quad \Delta_\lambda(a, b; q, t) = \prod_{1 \le j < k \le n} \frac{(qabt^{2n-j-k+1}; q)_{\lambda_j + \lambda_k} (t^{k-j+1}; q)_{\lambda_j - \lambda_k}}{(qabt^{2n-j-k}; q)_{\lambda_j + \lambda_k} (t^{k-j}; q)_{\lambda_j - \lambda_k}}, \qquad \forall \lambda \in \Lambda_n.
$$

Theorem 6.1. *For $\lambda \in \Lambda_n$ we have the evaluation formulas*

$$
(6.11) \quad P_\lambda^L(0) = \Delta_\lambda \prod_{i=1}^n \frac{(qat^{n-i}, qabt^{n-i}; q)_{\lambda_i}}{(qabt^{2(n-i)}; q)_{2\lambda_i}} (-1)^{\lambda_i} q^{\frac{1}{2}\lambda_i(\lambda_i - 1)},
$$

$$(6.12) \qquad P_\lambda^L(t^\rho) = \Delta_\lambda \prod_{i=1}^n \frac{\left(qbt^{n-i}, qabt^{n-i}; q\right)_{\lambda_i}}{\left(qabt^{2(n-i)}; q\right)_{2\lambda_i}} \left(aq^{\lambda_i}t^{n-i}\right)^{\lambda_i},$$

$$(6.13) \qquad P_\lambda^L(q^{-1}b^{-1}t^{-\rho}) = \Delta_\lambda \prod_{i=1}^n \frac{\left(qbt^{n-i}, qabt^{n-i}; q\right)_{\lambda_i}}{\left(qabt^{2(n-i)}; q\right)_{2\lambda_i}} \left(qbt^{n-i}\right)^{\lambda_i}.$$

Proof. The evaluation formulas depend rationally on the parameters, hence it suffices to prove the theorem for a suitable choice of Zariski dense complex values of the parameters a, b, q and t. The proof now uses the explicit limit transitions from Macdonald-Koornwinder polynomials to multivariable little q-Jacobi polynomials from [29] and [27]. The key ingredient is the following result from the proof of [27, Theorem 6.4]. Let $\lambda, \mu \in \Lambda_n$ such that $\mu < \lambda$. Then

$$(6.14) \qquad \lim_{\epsilon \to 0} \left(q^{-\frac{1}{2}}\epsilon\right)^{|\lambda|-|\mu|} c_{\lambda,\mu}\left(\epsilon^{-1}q^{\frac{1}{2}}, -aq^{\frac{1}{2}}, \epsilon bq^{\frac{1}{2}}, -q^{\frac{1}{2}}; q, t\right) = c_{\lambda,\mu}^L(a, b; q, t)$$

for the coefficients $c_{\lambda,\mu}$ (6.1) and $c_{\lambda,\mu}^L$ (2.1) in the monomial expansions of P_λ and P_λ^L, where $q^{\frac{1}{2}}$ is an arbitrary choice of square root of q.

Proof of (6.11). Observe that

$$(6.15) \qquad \lim_{\epsilon \to 0} \epsilon^{|\lambda|} m_\lambda(z) = \widetilde{m}_\lambda(0), \qquad \forall \lambda \in \Lambda_n,$$

since both sides are equal to zero if $\lambda \in \Lambda_n \setminus \{0\}$ and are equal to one if $\lambda = 0$. Combining (6.1), (2.1), (6.14) and (6.15) now yields

$$(6.16) \qquad \lim_{\epsilon \to 0} \left(q^{-\frac{1}{2}}\epsilon\right)^{|\lambda|} P_\lambda\left(z; \epsilon^{-1}q^{\frac{1}{2}}, -aq^{\frac{1}{2}}, \epsilon bq^{\frac{1}{2}}, -q^{\frac{1}{2}}; q, t\right) = P_\lambda^L(0; a, b; q, t)$$

for $\lambda \in \Lambda_n$. By the evaluation formula (6.2) for the Macdonald-Koornwinder polynomial P_λ and using that the Macdonald-Koornwinder polynomial is symmetric in a, b, c, d, we have

$$\left(q^{-\frac{1}{2}}\epsilon\right)^{|\lambda|} P_\lambda\left(-aq^{\frac{1}{2}}t^\rho; \epsilon^{-1}q^{\frac{1}{2}}, -aq^{\frac{1}{2}}, \epsilon bq^{\frac{1}{2}}, -q^{\frac{1}{2}}; q, t\right)$$
$$= \Delta_\lambda(a, b; q, t) \prod_{i=1}^n \frac{\left(qat^{n-i}, -qabt^{n-i}\epsilon, qabt^{n-i}, -qat^{n-i}\epsilon^{-1}; q\right)_{\lambda_i} \epsilon^{\lambda_i}}{\left(qabt^{2(n-i)}; q\right)_{2\lambda_i} \left(-aqt^{n-i}\right)^{\lambda_i}}.$$

Combined with the limit transition (6.16), specialized to $z = -aq^{\frac{1}{2}}t^\rho$, we obtain the evaluation formula (6.11) for the multivariable little q-Jacobi polynomial P_λ^L.

Proof of (6.12). Now we use the degeneration

$$(6.17) \qquad \lim_{\epsilon \to 0} \epsilon^{|\lambda|} m_\lambda(\epsilon^{-1}z) = \widetilde{m}_\lambda(z), \qquad \forall \lambda \in \Lambda_n$$

for the monomial bases. Together with (6.14) this yields the limit transition

$$(6.18) \qquad \lim_{\epsilon \to 0} \left(q^{-\frac{1}{2}}\epsilon\right)^{|\lambda|} P_\lambda\left(q^{\frac{1}{2}}\epsilon^{-1}z; \epsilon^{-1}q^{\frac{1}{2}}, -aq^{\frac{1}{2}}, \epsilon bq^{\frac{1}{2}}, -q^{\frac{1}{2}}; q, t\right) = P_\lambda^L(z; a, b; q, t)$$

for $\lambda \in \Lambda_n$, which is in accordance with [27, Theorem 6.4]. By the evaluation formula (6.2) for the Macdonald-Koornwinder polynomial P_λ we now have

$$
\left(q^{-\frac{1}{2}}\epsilon\right)^{|\lambda|} P_\lambda\left(q^{\frac{1}{2}}\epsilon^{-1}t^\rho; \epsilon^{-1}q^{\frac{1}{2}}, -aq^{\frac{1}{2}}, \epsilon bq^{\frac{1}{2}}, -q^{\frac{1}{2}}; q, t\right)
$$
$$
= \Delta_\lambda(a, b; q, t) \prod_{i=1}^{n} \frac{\left(-qat^{n-i}\epsilon^{-1}, -qt^{n-i}\epsilon^{-1}, qbt^{n-i}, qabt^{n-i}; q\right)_{\lambda_i} \epsilon^{2\lambda_i}}{\left(qabt^{2(n-i)}; q\right)_{2\lambda_i}\left(qt^{n-i}\right)^{\lambda_i}}.
$$

Combined with the limit transition (6.18), specialized to $z = t^\rho$, we obtain the evaluation formula (6.12).

Proof of (6.13). We now use the limit transition

$$(6.19) \qquad \lim_{\epsilon \to 0} \epsilon^{|\lambda|} m_\lambda(\epsilon z) = \widetilde{m}_\lambda(z^{-1}), \qquad \forall\, \lambda \in \Lambda_n.$$

Together with (6.14) this yields

$$(6.20) \qquad \lim_{\epsilon \to 0}\left(q^{-\frac{1}{2}}\epsilon\right)^{|\lambda|} P_\lambda\left(q^{-\frac{1}{2}}\epsilon z; \epsilon^{-1}q^{\frac{1}{2}}, -aq^{\frac{1}{2}}, \epsilon bq^{\frac{1}{2}}, -q^{\frac{1}{2}}; q, t\right) = P_\lambda^L(z^{-1}; a, b; q, t)$$

for $\lambda \in \Lambda_n$. By the evaluation formula (6.2) for the Macdonald-Koornwinder polynomial P_λ we now have

$$
\left(q^{-\frac{1}{2}}\epsilon\right)^{|\lambda|} P_\lambda\left(\epsilon bq^{\frac{1}{2}}t^\rho; \epsilon^{-1}q^{\frac{1}{2}}, -aq^{\frac{1}{2}}, \epsilon bq^{\frac{1}{2}}, -q^{\frac{1}{2}}; q, t\right)
$$
$$
= \Delta_\lambda(a, b; q, t) \prod_{i=1}^{n} \frac{\left(qbt^{n-i}, qabt^{n-i}, -qbt^{n-i}\epsilon, -qabt^{n-i}\epsilon; q\right)_{\lambda_i}}{\left(qabt^{2(n-i)}; q\right)_{2\lambda_i}}\left(qbt^{n-i}\right)^{-\lambda_i}.
$$

Combined with the limit transition (6.20), specialized to $z = qbt^\rho$, we obtain (6.13). $\square$

The quadratic norms $N_L(\lambda) = N_L(\lambda; a, b; q, t) \in \mathbb{E}$ of the multivariable little q-Jacobi polynomials have been explicitly evaluated in [27]. It reads

$$N_L(\lambda) = q^{(\lambda,\lambda)} a^{|\lambda|} t^{2(\rho,\lambda)} N_L^+(\lambda) N_L^-(\lambda),$$

where the factors $N_L^\pm(\lambda) = N_L^\pm(\lambda; a, b; q, t) \in \mathbb{E}$ are given by

$$
N_L^+(\lambda) = \prod_{i=1}^{n} \frac{\left(qat^{n-i}, qabt^{n-i}; q\right)_{\lambda_i}}{\left(qabt^{2(n-i)}; q\right)_{2\lambda_i}} \prod_{1 \le j < k \le n} \frac{\left(qabt^{2n-j-k+1}; q\right)_{\lambda_j+\lambda_k}\left(t^{k-j+1}; q\right)_{\lambda_j-\lambda_k}}{\left(qabt^{2n-j-k}; q\right)_{\lambda_j+\lambda_k}\left(t^{k-j}; q\right)_{\lambda_j-\lambda_k}},
$$
$$
N_L^-(\lambda) = \prod_{i=1}^{n} \frac{\left(qt^{n-i}, qbt^{n-i}; q\right)_{\lambda_i}}{\left(q^2abt^{2(n-i)}; q\right)_{2\lambda_i}} \prod_{1 \le j < k \le n} \frac{\left(q^2abt^{2n-j-k-1}; q\right)_{\lambda_j+\lambda_k}\left(qt^{k-j-1}; q\right)_{\lambda_j-\lambda_k}}{\left(q^2abt^{2n-j-k}; q\right)_{\lambda_j+\lambda_k}\left(qt^{k-j}; q\right)_{\lambda_j-\lambda_k}}.
$$

We recall from [27] that

$$(6.21) \quad \lim_{\epsilon \to 0}\left(q^{-\frac{1}{2}}\epsilon\right)^{2|\lambda|} N_K\left(\lambda; \epsilon^{-1}q^{\frac{1}{2}}, -aq^{\frac{1}{2}}, \epsilon bq^{\frac{1}{2}}, -q^{\frac{1}{2}}; q, t\right) = N_L(\lambda; a, b; q, t), \qquad \forall\, \lambda \in \Lambda_n$$

for generic specialized parameters.

6.3. **Multivariable big q-Jacobi polynomials.** We repeat the techniques and arguments of the previous subsection to derive evaluation formulas for multivariable big q-Jacobi polynomials. The monic multivariable big q-Jacobi polynomial

$$P_\lambda^B(z) = P_\lambda^B(z; a, b, c, d; q, t) \in \mathbb{E}[z]^S$$

of degree $\lambda \in \Lambda_n$ is of the form

$$(6.22) \qquad P_\lambda^B(z) = \widetilde{m}_\lambda(z) + \sum_{\mu \in \Lambda : \mu < \lambda} c_{\lambda,\mu}^B \widetilde{m}_\mu(z)$$

for certain coefficients $c_{\lambda,\mu}^B = c_{\lambda,\mu}^B(a, b, c, d; q, t) \in \mathbb{E}$. Similarly as for the Koornwinder polynomials and the multivariable little q-Jacobi polynomials, the multivariable big q-Jacobi polynomials can be characterized by a second-order difference equation they should satisfy, or in terms of suitable orthogonality properties, see [26]. The characterization as solution of a difference equation is $D_B P_\lambda^B = E_\lambda^{qJ} P_\lambda^B$ with

$$D_B = \sum_{j=1}^n \left(\phi_{B,j}^+(z)(T_j - \mathrm{Id}) + \phi_{B,j}^-(z)(T_j^{-1} - \mathrm{Id}) \right),$$

$$\phi_{B,j}^+(z) = qt^{n-1}\left(a - \frac{c}{qz_j}\right)\left(b + \frac{d}{qz_j}\right) \prod_{l \neq j} \frac{z_l - tz_j}{z_l - z_j},$$

$$\phi_{B,j}^-(z) = \left(1 - \frac{c}{z_j}\right)\left(1 + \frac{d}{z_j}\right) \prod_{l \neq j} \frac{z_j - tz_l}{z_j - z_l}.$$

We again consider the evaluation formulas for the one-variable big q-Jacobi polynomials ($n = 1$) first. The one-variable monic big q-Jacobi polynomial is independent of t and is given explicitly as

$$(6.23) \qquad P_m^B(z) = \frac{(qa, -qad/c; q)_m}{(q^{m+1}ab; q)_m (qa/c)^m} {}_3\phi_2\left(\begin{array}{c} q^{-m}, q^{m+1}ab, qaz/c \\ qa, -qad/c \end{array}; q, q\right)$$

for $m \in \mathbb{Z}_+$, see e.g. [14]. We immediately obtain from (6.23) the evaluation formula

$$(6.24) \qquad P_m^B(c/qa) = \frac{(qa, -qad/c; q)_m}{(q^{m+1}ab; q)_m}(c/qa)^m$$

for $m \in \mathbb{Z}_+$. A straightforward computation using (6.23) and the q-Vandermonde formula (6.7) yields the two evaluation formulas

$$(6.25) \qquad \begin{aligned} P_m^B(c) &= \frac{(qa, qab, -qbc/d; q)_m}{(qab; q)_{2m}} d^m q^{\frac{1}{2}m(m-1)}, \\ P_m^B(-d) &= \frac{(-qad/c, qb, qab; q)_m}{(qab; q)_{2m}}(-1)^m c^m q^{\frac{1}{2}m(m-1)} \end{aligned}$$

for $m \in \mathbb{Z}_+$. Finally, by the q-Saalschütz sum

$$(6.26) \qquad {}_3\phi_2 \begin{pmatrix} q^{-m}, a, b \\ abq^{1-m}/c, c \end{pmatrix}; q, q \end{pmatrix} = \frac{\big(c/a, c/b; q\big)_m}{\big(c, c/ab; q\big)_m}, \qquad m \in \mathbb{Z}_+$$

(see [8, (II.12)]) we obtain the evaluation formula

$$(6.27) \qquad P_m^B(-d/qb) = \frac{\big(qb, -qbc/d, qab; q\big)_m}{\big(qab; q\big)_{2m}} \big(-d/qb\big)^m$$

for $m \in \mathbb{Z}_+$. In the following theorem we give the multivariable analogues of the evaluation formulas (6.24), (6.25) and (6.27). Recall the explicit expression Δ_λ, defined by (6.10).

Theorem 6.2. *For $\lambda \in \Lambda_n$ we have the evaluation formulas*

$$(6.28) \qquad P_\lambda^B(ct^\rho) = \Delta_\lambda \prod_{i=1}^n \frac{\big(qat^{n-i}, qabt^{n-i}, -\frac{qbc}{d}t^{n-i}; q\big)_{\lambda_i}}{\big(qabt^{2(n-i)}; q\big)_{2\lambda_i}} d^{\lambda_i} q^{\frac{1}{2}\lambda_i(\lambda_i - 1)},$$

$$(6.29) \qquad P_\lambda^B(-dt^\rho) = \Delta_\lambda \prod_{i=1}^n \frac{\big(qbt^{n-i}, qabt^{n-i}, -\frac{qad}{c}t^{n-i}; q\big)_{\lambda_i}}{\big(qabt^{2(n-i)}; q\big)_{2\lambda_i}} (-c)^{\lambda_i} q^{\frac{1}{2}\lambda_i(\lambda_i - 1)},$$

$$(6.30) \qquad P_\lambda^B\big(\frac{c}{qa} t^{-\rho}\big) = \Delta_\lambda \prod_{i=1}^n \frac{\big(qat^{n-i}, qabt^{n-i}, -\frac{qad}{c}t^{n-i}; q\big)_{\lambda_i}}{\big(qabt^{2(n-i)}; q\big)_{2\lambda_i}} \big(\frac{c}{qa} t^{i-n}\big)^{\lambda_i},$$

$$(6.31) \qquad P_\lambda^B\big(-\frac{d}{qb} t^{-\rho}\big) = \Delta_\lambda \prod_{i=1}^n \frac{\big(qbt^{n-i}, qabt^{n-i}, -\frac{qbc}{d}t^{n-i}; q\big)_{\lambda_i}}{\big(qabt^{2(n-i)}; q\big)_{2\lambda_i}} \big(-\frac{d}{qb} t^{i-n}\big)^{\lambda_i}.$$

Proof. The proof is analogous to the proof of Theorem 6.2, so we only indicate the main steps. From (the proof of) [27, Theorem 7.5] (replacing the role of the limiting parameter ϵ in [27, Theorem 7.5] by $\big(cd/q\big)^{\frac{1}{2}}\epsilon$), we obtain the two limit transitions

$$(6.32) \qquad \lim_{\epsilon \to 0} \epsilon^{|\lambda|} P_\lambda\Big(\epsilon^{-1}z; \frac{qa\epsilon}{c}, -\frac{qb\epsilon}{d}, c\epsilon^{-1}, -d\epsilon^{-1}; q, t\Big) = P_\lambda^B(z; a, b, c, d; q, t),$$

$$(6.33) \qquad \lim_{\epsilon \to 0} \epsilon^{|\lambda|} P_\lambda\Big(\epsilon z; \frac{qa\epsilon}{c}, -\frac{qb\epsilon}{d}, c\epsilon^{-1}, -d\epsilon^{-1}; q, t\Big) = P_\lambda^B(z^{-1}; a, b, c, d; q, t)$$

for $\lambda \in \Lambda_n$ and for a specific Zariski-dense subset of specialized parameters a, b, c, d, q and t. If we set $z = \epsilon^{-1}ct^\rho$ (respectively $z = -\epsilon^{-1}dt^\rho$) in (6.32) and use the evaluation formula (6.2) for the Macdonald-Koornwinder polynomial P_λ, then we arrive at the evaluation formula (6.28) (respectively (6.29)). The evaluation formulas (6.30) and (6.31) follow from (6.2) and the limit transition (6.33) specialized to $z = \frac{qa}{c} t^\rho$ and $z = -\frac{qb}{d} t^\rho$, respectively. $\qquad\square$

We end this section by recalling the quadratic norm evaluations of the multivariable big q-Jacobi polynomials from [27]. We define an $\mathbb{E}$-linear functional

$$h_B = h_B^{a,b,c,d;q,t} : \mathbb{E}[z]^S \to \mathbb{E}$$

by requiring

$$h_B\big(P_\lambda^B\big) = \begin{cases} 1, & \text{if } \lambda = 0, \\ 0, & \text{if } \lambda \neq 0. \end{cases}$$

An analytic definition of h_B can be given for specific Zariski dense choice of specialized parameters as a multidimensional q-integral with explicit weight function, see e.g. [27]. We define a sesqui-linear form $\langle \cdot, \cdot \rangle_B = \langle \cdot, \cdot \rangle_B^{a,b,c,d;q,t}$ on $\mathbb{E}[z]^S$ by

(6.34) $$\langle p_1, p_2 \rangle_B = h_B\big(p_1(z)\overline{p_2(z)}\big), \qquad p_1, p_2 \in \mathbb{E}[z]^S,$$

with the same convention on the complex conjugation as for the multivariable little q-Jacobi polynomials. The orthogonality relations for the multivariable big q-Jacobi polynomials are now given by

(6.35) $$\langle P_\lambda^B, P_\mu^B \rangle_B = N_B(\lambda)\delta_{\lambda,\mu}, \qquad \forall \lambda, \mu \in \Lambda_n$$

with the quadratic norms $N_B(\lambda) = N_B(\lambda; a, b, c, d; q, t) \in \mathbb{E}$ given by

$$N_B(\lambda) = (cd)^{|\lambda|} t^{\langle \rho, \lambda \rangle} \left(\prod_{i=1}^n q^{\binom{\lambda_i}{2}} \Big(-\frac{qbc}{d} t^{n-i}, -\frac{qad}{c} t^{n-i}; q\Big)_{\lambda_i} \right) N_L^+(\lambda) N_L^-(\lambda),$$

with $N_L^\pm = N_L^\pm(a, b; q, t)$ the same factors as for the multivariable little q-Jacobi polynomials. We recall from [27] that

(6.36) $$\lim_{\epsilon \to 0} \epsilon^{2|\lambda|} N_K\big(\lambda; \frac{qa\epsilon}{c}, -\frac{qb\epsilon}{d}, c\epsilon^{-1}, -d\epsilon^{-1}; q, t\big) = N_B(\lambda; a, b, c, d; q, t), \qquad \lambda \in \Lambda_n,$$

for generic specialized parameters.

7. Generalized dimension formulas

In this section we consider the generalized dimensions $D_q(\lambda; a, b; t)$ (see (2.4)) in detail. We suppress the dependence on the parameters a, b and t as much as possible, so we write $D_q(\lambda) = D_q(\lambda; a, b; t)$. We will show in Section 8 that $D_q(\lambda)$ for specialized parameters gives the quantum dimension for the irreducible spherical representation V_λ^q ($\lambda \in \Lambda_n$) associated to the standard quantum complex Grassmannian. The degeneration ($q = 1$) of the quantum dimensions yields the complex dimensions of the irreducible $\mathcal{M}$-spherical $\mathcal{K}$-representations associated to the complex and real Grassmannian, see Subsection 4.2. The degeneration ($q = 0$), which corresponds to the dimensions of the irreducible $\mathcal{M}_\mathbb{F}$-spherical $\mathcal{K}_\mathbb{F}$-representations $V_\lambda^\mathbb{F}$ for $\mathbb{F}$ non-Archimedean (see Subsection 4.3) is analyzed in detail in Subsection 7.2.

7.1. **Generalized quantum dimensions.** We begin by giving an explicit expression for the generalized dimension $D_q(\lambda)$.

Lemma 7.1. *We have*

$$D_q(\lambda) = a^{-|\lambda|} t^{-2(\rho,\lambda)} \prod_{i=1}^n v_i(\lambda_i; q) \prod_{1 \leq j < k \leq n} w_{j,k}^+(\lambda_j + \lambda_k; q) w_{j,k}^-(\lambda_j - \lambda_k; q), \qquad \lambda \in \Lambda_n,$$

where $v_i(m; q)$, $w_{j,k}^{\pm}(m; q)$ for $m \in \mathbb{Z}_+$ are given by

$$v_i(m; q) = \frac{\left(at^{n-i}, q^{-1}abt^{n-i}; q\right)_m}{\left(qt^{n-i}, bt^{n-i}; q\right)_m} \frac{(1 - q^{2m-1}abt^{2(n-i)})}{(1 - q^{-1}abt^{2(n-i)})},$$

$$w_{j,k}^+(m; q) = \frac{\left(q^{-1}abt^{2n-j-k+1}; q\right)_m}{\left(abt^{2n-j-k-1}; q\right)_m} \frac{(1 - q^{m-1}abt^{2n-j-k})}{(1 - q^{-1}abt^{2n-j-k})},$$

$$w_{j,k}^-(m; q) = \frac{\left(t^{k-j+1}; q\right)_m}{\left(qt^{k-j-1}; q\right)_m} \frac{(1 - q^m t^{k-j})}{(1 - t^{k-j})}.$$

Proof. This follows from a direct computation using the evaluation formulas and the quadratic norm formulas for the multivariable little q-Jacobi polynomials. $\square$

The explicit expressions for the generalized dimensions associated to the fundamental partitions $\omega_r \in \Lambda_n$ $(r = 0, \ldots, n)$ simplify drastically.

Proposition 7.2. *For $r = 0, \ldots, n$ we have*

$$(7.1) \quad D_q(\omega_r) = \frac{\left(qabt^{2n-r-1}, t^{n+1-r}, at^{n-r}, abt^{2n-r}; t\right)_r}{\left(q, t, bt^{n-r}, abt^{n-r-1}; t\right)_r} \left(\frac{1 - abt^{2n-2r-1}}{1 - abt^{2n-1}}\right) a^{-r} t^{r(r+1-2n)}.$$

Proof. Note that

$$|\omega_r| = r, \qquad 2(\rho, \omega_r) = r(2n - r - 1).$$

By Lemma 7.1 we thus obtain
(7.2)

$$D_q(\omega_r) = a^{-r} t^{r(r+1-2n)} \prod_{i=1}^r v_i(1; q) \prod_{1 \leq j < k \leq r} w_{j,k}^+(2; q) \prod_{j=1}^r \prod_{k=r+1}^n w_{j,k}^+(1; q) \prod_{j=1}^r \prod_{k=r+1}^n w_{j,k}^-(1; q).$$

By direct computations we obtain the explicit expressions

$$
(7.3) \quad
\begin{aligned}
\prod_{i=1}^{r} v_i(1;q) &= \frac{\left(at^{n-r}, q^{-1}abt^{n-r};t\right)_r \left(qabt^{2n-2r};t^2\right)_r}{\left(qt^{n-r}, bt^{n-r};t\right)_r \left(q^{-1}abt^{2n-2r};t^2\right)_r}, \\
\prod_{1\leq j<k\leq r} w_{j,k}^+(2;q) &= \frac{\left(abt^{2n-r};t\right)_r \left(q^{-1}abt^{2n-2r}, qabt^{2n-2r-1};t^2\right)_r}{\left(q^{-1}abt^{2n-2r}, qabt^{2n-2r-1}, abt^{2n-2r-1};t\right)_r} \frac{(1-abt^{2n-2r-1})}{(1-abt^{2n-1})}, \\
\prod_{j=1}^{r}\prod_{k=r+1}^{n} w_{j,k}^+(1;q) &= \frac{\left(q^{-1}abt^{2n-2r}, abt^{2n-2r-1};t\right)_r}{\left(q^{-1}abt^{n-r}, abt^{n-r-1};t\right)_r}, \\
\prod_{j=1}^{r}\prod_{k=r+1}^{n} w_{j,k}^-(1;q) &= \frac{\left(t^{n+1-r}, qt^{n-r};t\right)_r}{\left(t, q;t\right)_r}.
\end{aligned}
$$

Only the second formula needs explanation. Set

$$
e_r(u;t) = \prod_{1\leq j<k\leq r}\left(\frac{1-ut^{1-j-k}}{1-ut^{-j-k}}\right),
$$

then we can write

$$
\prod_{1\leq j<k\leq r} w_{j,k}^+(2;q) = e_r(q^{-1}abt^{2n})e_r(abt^{2n})e_r(abt^{2n-1})e_r(qabt^{2n-1}).
$$

On the other hand, by induction to $r \in \{1,\ldots,n\}$,

$$
e_r(u;t) = \frac{\left(ut^{2-2r};t^2\right)_{r-1}}{\left(ut^{1-2r};t\right)_{r-1}}.
$$

Combining both formulas we easily obtain the desired expression for $\prod_{1\leq j<k\leq r} w_{j,k}^+(2;q)$. The lemma now follows by substituting the expressions (7.3) in (7.2), and by simplifying the resulting expression. $\qquad\square$

As a curiosity, note that the elementary identity

$$
\left(ut^{1-r};t\right)_r = (-u)^r t^{-\frac{1}{2}r(r-1)}\left(u^{-1};t\right)_r
$$

allows us to rewrite (7.1) as

$$
D_q(\omega_r) = \frac{\left(t^{-n}, a^{-1}t^{1-n}, a^{-1}b^{-1}t^{1-2n}, q^{-1}a^{-1}b^{-1}t^{2-2n};t\right)_r}{\left(t, q, b^{-1}t^{1-n}, a^{-1}b^{-1}t^{2-n};t\right)_r}\left(\frac{1-a^{-1}b^{-1}t^{1+2r-2n}}{1-a^{-1}b^{-1}t^{1-2n}}\right)\left(qat^{2n-1}\right)^r
$$

for $r = 0,\ldots,n$, which one recognizes as the weight function for t-Racah polynomials, see e.g. [14].

7.2. **Generalized p-adic dimensions.** In this subsection we consider the p-adic degeneration of the generalized quantum dimension $D_q(\lambda)$ and we relate it to the complex dimension of the $\mathcal{M}_{\mathbb{F}}$-spherical irreducible $\mathcal{K}_{\mathbb{F}}$-representation $V_\lambda^{\mathbb{F}}$ for non-Archimedean local fields $\mathbb{F}$ (see Section 4). Denote $\delta_{k,l}$ $(k, l \in \mathbb{Z})$ for the Kronecker delta function: it is one if $k = l$ and zero otherwise.

Lemma 7.3. *The functions $v_i(m; q)$, $w_{j,k}^{\pm}(m; q)$ (see Lemma 7.1) for $m \in \mathbb{Z}_+$ are regular at $q = 0$. If we write $v_i(m) = v_i(m; 0)$ and $w_{j,k}^{\pm}(m) = w_{j,k}^{\pm}(m; 0)$ for their constant terms, then $v_i(0) = w_{j,k}^{\pm}(0) = 1$ and*

$$v_i(m) = \left(\frac{1 - at^{n-i}}{1 - bt^{n-i}} \right) t^{i-n} \left(1 - (1 - \delta_{m,1}) abt^{n-i} \right),$$

$$w_{j,k}^{+}(m) = \left(\frac{1 - abt^{2n-j-k+1-\delta_{m,1}}}{1 - abt^{2n-j-k-1}} \right) t,$$

$$w_{j,k}^{-}(m) = \frac{1 - t^{k-j+1}}{1 - t^{k-j}}$$

for $m \geq 1$.

Proof. As an example, we compute $v_i(m)$ for $m \in \mathbb{Z}_{\geq 1}$ (the other computations are similar). To compute $v_i(1)$ we rewrite $v_i(1; q)$ as

$$v_i(1; q) = \left(\frac{1 - qabt^{2(n-i)}}{1 - qt^{n-i}} \right) \left(\frac{1 - at^{n-i}}{1 - bt^{n-i}} \right) \left(\frac{q - abt^{n-i}}{q - abt^{2(n-i)}} \right),$$

which yields

$$v_i(1) = v_i(1; 0) = \left(\frac{1 - at^{n-i}}{1 - bt^{n-i}} \right) t^{i-n}.$$

To compute $v_i(m)$ for $m \geq 2$ we write $v_i(m; q)$ as

$$v_i(m; q) = \left(\frac{(1 - q^{2m-1} abt^{2(n-i)}) (qat^{n-i}; q)_{m-1} (qabt^{n-i}; q)_{m-2}}{(qt^{n-i}; q)_m (qbt^{n-i}; q)_{m-1}} \right)$$
$$\times \left(\frac{(1 - at^{n-i})(1 - abt^{n-i})}{(1 - bt^{n-i})} \right) \left(\frac{q - abt^{n-i}}{q - abt^{2(n-i)}} \right),$$

which yields

$$v_i(m) = v_i(m; 0) = \left(\frac{1 - at^{n-i}}{1 - bt^{n-i}} \right) t^{i-n} (1 - abt^{n-i}), \qquad m \geq 2.$$

$\square$

Proposition 7.4. *Let $\lambda \in \Lambda_n$. The generalized quantum dimension $D_q(\lambda)$ (see (2.4)) is regular at $q = 0$. Writing $D_0(\lambda) = D_0(\lambda; a, b; t)$ for $D_q(\lambda; a, b; t)|_{q=0}$, we have*

$$D_0(\lambda) = a^{-|\lambda|} t^{-2(\rho, \lambda)} \prod_{i=1}^{n} v_i(\lambda_i) \prod_{1 \leq j < k \leq n} w_{j,k}^{+}(\lambda_j + \lambda_k) w_{j,k}^{-}(\lambda_j - \lambda_k), \qquad \lambda \in \Lambda_n.$$

Proof. This is immediate from Lemma 7.1 and the previous lemma.   □

Example 7.5. *For $n = 1$, so that $\Lambda_1 = \mathbb{Z}_+$, we have $D_0(0) = 1$ and*

$$D_0(\lambda) = a^{-\lambda} \left(\frac{1-a}{1-b} \right) \left(1 - (1 - \delta_{\lambda,1})ab \right), \qquad \lambda \geq 1.$$

In contrast to the quantum case, the explicit expression for $D_0(\lambda) = D_0(\lambda; a, b; t)$ from Proposition 7.4 simplifies for all partitions $\lambda \in \Lambda_n$ as follows.

Theorem 7.6. *For $\lambda \in \Lambda_n$ we have*

$$(7.4) \quad D_0(\lambda) = a^{-|\lambda|} t^{-2(\rho,\lambda)} \begin{bmatrix} n \\ \partial\lambda' \end{bmatrix}_t \frac{\left(at^{n-\lambda'_1}; t \right)_{\lambda'_1} \left(abt^{2n-\lambda'_1-\lambda'_2}; t \right)_{\lambda'_1+\lambda'_2}}{\left(bt^{n-\lambda'_1}, abt^{n-\lambda'_1-1}; t \right)_{\lambda'_1}} \left(\frac{1 - abt^{2n-2\lambda'_1-1}}{1 - abt^{2n-1}} \right).$$

As remarked already in Subsection 4.3, the generalized p-adic dimension formula (7.4) combined with its representation theoretic interpretation (4.14) from [23] leads to the explicit dimension formulas (4.13) for the irreducible $\mathcal{M}_{\mathbb{F}}$-spherical $\mathcal{K}_{\mathbb{F}}$-representations $V_\lambda^{\mathbb{F}}$ ($\lambda \in \Lambda_n$).

Corollary 7.7. *For $r = 0, \ldots, n$ we have*

$$(7.5) \qquad D_0(\omega_r) = \frac{\left(t^{n+1-r}, at^{n-r}, abt^{2n-r}; t \right)_r}{\left(t, bt^{n-r}, abt^{n-r-1}; t \right)_r} \left(\frac{1 - abt^{2n-2r-1}}{1 - abt^{2n-1}} \right) a^{-r} t^{r(r+1-2n)}.$$

Proof. Follows either by specializing Theorem 7.6 to $\lambda = \omega_r$, or by taking $q = 0$ in (7.1).   □

We next proceed to prove Theorem 7.6, which is based on the expression for $D_0(\lambda)$ as given in Proposition 7.4. We divide the proof into several elementary lemmas. Since the theorem is obviously correct for $\lambda = 0$, we fix for the proof a nonzero partition $\lambda \in \Lambda_n$.

Lemma 7.8.

$$\prod_{i=1}^n v_i(\lambda_i) = t^{\frac{1}{2}\lambda'_1(\lambda'_1+1)-n\lambda'_1} \frac{\left(at^{n-\lambda'_1}; t \right)_{\lambda'_1} \left(abt^{n-\lambda'_2}; t \right)_{\lambda'_2}}{\left(bt^{n-\lambda'_1}; t \right)_{\lambda'_1}}.$$

Proof. Using the explicit expression for v_i (see Lemma 7.3), we have

$$\prod_{i=1}^n v_i(\lambda_i) = \prod_{i=1}^{\lambda'_2} \left(\frac{1 - at^{n-i}}{1 - bt^{n-i}} \right) t^{i-n} (1 - abt^{n-i}) \prod_{i=\lambda'_2+1}^{\lambda'_1} \left(\frac{1 - at^{n-i}}{1 - bt^{n-i}} \right) t^{i-n}$$

$$= \prod_{i=1}^{\lambda'_1} \left(\frac{1 - at^{n-i}}{1 - bt^{n-i}} \right) t^{i-n} \prod_{i=1}^{\lambda'_2} \left(1 - abt^{n-i} \right)$$

$$= t^{\frac{1}{2}\lambda'_1(\lambda'_1+1)-n\lambda'_1} \frac{\left(at^{n-\lambda'_1}; t \right)_{\lambda'_1} \left(abt^{n-\lambda'_2}; t \right)_{\lambda'_2}}{\left(bt^{n-\lambda'_1}; t \right)_{\lambda'_1}},$$

as desired.   □

Lemma 7.9.

$$\prod_{1 \leq j < k \leq n} w_{j,k}^-(\lambda_j - \lambda_k) = \begin{bmatrix} n \\ \partial \lambda' \end{bmatrix}_t.$$

Proof. Since $w_{j,k}^-(0) = 1$ and $w_{j,k}^-(m) = \frac{1-t^{k-j+1}}{1-t^{k-j}}$ for $m \neq 0$ (see Lemma 7.3), the term $w_{j,k}^-(\lambda_j - \lambda_k)$ contributes to the product only if $\lambda_j \neq \lambda_k$. Hence

$$\prod_{1 \leq j < k \leq n} w_{j,k}^-(\lambda_j - \lambda_k) = A/B,$$

with

$$A = \prod_{1 \leq j < k \leq n} \frac{1 - t^{k-j+1}}{1 - t^{k-j}} = \prod_{l=1}^n \frac{(1 - t^{l+1})^{n-l}}{(1 - t^l)^{n-l}} = [n]_t!$$

$$B = \prod_{\substack{1 \leq j < k \leq n \\ \lambda_j = \lambda_k}} \frac{1 - t^{k-j+1}}{1 - t^{k-j}} = \prod_{l \geq 0} \prod_{1 \leq j < k \leq \lambda'_l - \lambda'_{l+1}} \frac{1 - t^{k-j+1}}{1 - t^{k-j}} = \prod_{l \geq 0} [\lambda'_l - \lambda'_{l+1}]_t!,$$

as desired. $\square$

Lemma 7.10.

$$\prod_{1 \leq j < k \leq n} w_{j,k}^+(\lambda_j + \lambda_k) = t^{\frac{1}{2}\lambda'_1(\lambda'_1 - 1) + \lambda'_1(n - \lambda'_1)} \frac{(abt^{2n-\lambda'_1}; t)_{\lambda'_1 - 1}}{(abt^{2n-2\lambda'_1}; t)_{\lambda'_1 - 1}} \frac{(abt^{2n-2\lambda'_1 - 1}; t)_{\lambda'_1} (abt^{2n-\lambda'_1 - \lambda'_2}; t)_{\lambda'_2}}{(abt^{n-\lambda'_1 - 1}; t)_{\lambda'_1} (abt^{n-\lambda'_2}; t)_{\lambda'_2}}.$$

Proof. The term $w_{j,k}^+(m)$ has three possible values depending on m being 0, 1 or ≥ 2 (see Lemma 7.3). We are hence led to consider the following sets

$$X_1 = \{(j,k) \,|\, 1 \leq j < k \leq n, \ \lambda_j + \lambda_k = 1\} = \{(j,k) \,|\, \lambda'_2 < j \leq \lambda'_1 < k \leq n\}$$
$$X_2 = \{(j,k) \,|\, 1 \leq j < k \leq n, \ \lambda_j + \lambda_k \geq 2\} = X_2^a \sqcup X_2^b,$$

where the disjoint subsets X_2^a and X_2^b are given by

$$X_2^a = \{(j,k) \,|\, 1 \leq j < k \leq \lambda'_1\}, \qquad X_2^b = \{(j,k) \,|\, 1 \leq j \leq \lambda'_2, \ \lambda'_1 < k \leq n\}.$$

We then have

$$\prod_{1 \leq j < k \leq n} w_{j,k}^+(\lambda_j + \lambda_k) = \Pi(X_1)\Pi(X_2^a)\Pi(X_2^b)$$

where we write $\Pi(X) = \prod_{(j,k) \in X} w_{j,k}^+(\lambda_j + \lambda_k)$ for a subset $X \subseteq \{(j,k) \,|\, 1 \leq j < k \leq n\}$. We have

$$\Pi(X_1) = t^{(n-\lambda'_1)(\lambda'_1 - \lambda'_2)} \prod_{j=\lambda'_2 + 1}^{\lambda'_1} \prod_{k=\lambda'_1 + 1}^{n} \left(\frac{1 - abt^{2n-j-k}}{1 - abt^{2n-j-k-1}} \right)$$

$$= t^{(n-\lambda'_1)(\lambda'_1 - \lambda'_2)} \prod_{j=\lambda'_2 + 1}^{\lambda'_1} \left(\frac{1 - abt^{2n-j-\lambda'_1 - 1}}{1 - abt^{n-j-1}} \right) = t^{(n-\lambda'_1)(\lambda'_1 - \lambda'_2)} \frac{(abt^{2n-2\lambda'_1 - 1}; t)_{\lambda'_1 - \lambda'_2}}{(abt^{n-\lambda'_1 - 1}; t)_{\lambda'_1 - \lambda'_2}}.$$

Using the same notations and simplification arguments as in the proof of Proposition 7.2, we have

$$\Pi(X_2^a) = t^{\frac{1}{2}\lambda_1'(\lambda_1'-1)} e_{\lambda_1'}(abt^{2n}) e_{\lambda_1'}(abt^{2n-1}) = t^{\frac{1}{2}\lambda_1'(\lambda_1'-1)} \frac{\left(abt^{2n-\lambda_1'}; t\right)_{\lambda_1'-1}}{\left(abt^{2n-2\lambda_1'}; t\right)_{\lambda_1'-1}}.$$

By a similar computation as for $\Pi(X_1)$, we finally have

$$\Pi(X_2^b) = t^{(n-\lambda_1')\lambda_2'} \frac{\left(abt^{2n-\lambda_1'-\lambda_2'}, abt^{2n-\lambda_1'-\lambda_2'-1}; t\right)_{\lambda_2'}}{\left(abt^{n-\lambda_2'}, abt^{n-\lambda_2'-1}; t\right)_{\lambda_2'}}.$$

Combining these expressions leads to the desired result. $\qquad\square$

We can now use Proposition 7.4, Lemma 7.8, Lemma 7.9 and Lemma 7.10 to obtain an explicit expression for the generalized p-adic dimension $D_0(\lambda)$. After straightforward simplifications, we arrive at the desired expression (7.4) for $D_0(\lambda)$.

8. The one-parameter family of quantum complex Grassmannians

In this section we consider the one-parameter family of quantum complex Grassmannians from [20], [5] and [22] (see also [17] for the general theory of harmonic analysis on quantum compact symmetric spaces). This more general set-up does no longer fit into the framework of quantum Gelfand pairs associated to CQG algebras as discussed in Section 3, since the role of the quantum subgroup is replaced by a suitable infinitesimal counterpart, depending on an additional continuous parameter.

In [20] and [5] harmonic analysis on the one-parameter family of quantum complex Grassmannians was studied in detail, leading to the interpretation of Macdonald-Koornwinder polynomials and multivariable big and little q-Jacobi polynomials as the associated quantum zonal spherical functions. From these results we now derive representation theoretic interpretations of the evaluation formulas and norm formulas for Macdonald-Koornwinder polynomials and multivariable big and little q-Jacobi polynomials. The arguments follow closely [19, Section 6], now applied to the one-parameter family of quantum complex Grassmannians. In this more general context we discuss the quantum dimension formulas for irreducible spherical representations in terms of quantum zonal spherical functions. The analysis leads to the quantum dimension formulas (5.5) and (5.6) for the irreducible $\mathcal{U}_q$-representations V_λ^q ($\lambda \in \Lambda_n$).

We now first recall the construction of the quantum zonal spherical functions on a one-parameter family of quantum analogues of the complex Grassmannian $\mathcal{K}/\mathcal{M}$. We will freely use the notations from Section 5. The role of the subgroup $\mathcal{M}$ is taken over by a two-sided co-ideal $\mathfrak{k}_\sigma \subseteq \mathcal{U}_q$ depending (in a suitable sense continuously) on an additional parameter $\sigma \in \mathbb{R} \cup \{\infty\}$, see e.g. [20], [5] for the explicit definition of $\mathfrak{k}_\sigma$. The case $\sigma = \infty$ corresponds to the standard quantum complex Grassmannian from Section 5.

The space of $\mathfrak{k}_\sigma$-invariants in $\mathbb{C}_q(\mathcal{K})$,

$$\mathbb{C}_q^\sigma(\mathcal{K}/\mathcal{M}) := \{f \in \mathbb{C}_q(\mathcal{K}) \mid f(YX) = 0, \qquad \forall X \in \mathfrak{k}_\sigma, \ \forall Y \in \mathcal{U}_q\}$$

is a $\mathcal{U}_q$-submodule of $\mathbb{C}_q(\mathcal{K})$ with respect to the regular $\mathcal{U}_q$-action

$$\left(X\,f\right)(X') = f(S(X)X'), \qquad f \in \mathbb{C}_q^\sigma(\mathcal{K}/\mathcal{M}), \quad X, X' \in \mathcal{U}_q.$$

The $\mathcal{U}_q$-module $\mathbb{C}_q^\sigma(\mathcal{K}/\mathcal{M})$ serves as a one-parameter family of quantum analogues of the Grassmann representation $\mathbb{C}(\mathcal{K}/\mathcal{M})$. Furthermore, $\mathbb{C}_q^\infty(\mathcal{K}/\mathcal{M})$ is the Grassmann representation $\mathbb{C}_q(\mathcal{K}/\mathcal{M})$ associated to the standard quantum complex Grassmannian from Section 5. The space

$$\left(L_\mu^q\right)^{\mathfrak{k}_\sigma} = \{v \in L_\mu^q \mid Xv = 0, \quad \forall\, X \in \mathfrak{k}_\sigma\}$$

for $\mu \in P_d^+$ serves as the one-parameter quantum analogue of the space of $\mathcal{M}$-fixed vectors in $L_\mu^{\mathbb{C}}$. The subspace $\left(L_\mu^q\right)^{\mathfrak{k}_\sigma}$ is one-dimensional or zero dimensional, and it is one-dimensional if and only if $\mu \in \Lambda_n^\natural$. Consequently, the irreducible spherical representations $V_\lambda^q = L_{\lambda^\natural}^q$ ($\lambda \in \Lambda_n$) associated to the standard quantum complex Grassmannian also serve as complete set of representatives of finite dimensional, irreducible $\mathcal{U}_q$-representations with nonzero (in fact, one-dimensional) space of $\mathfrak{k}_\sigma$-invariants. From the Peter-Weyl decomposition for $\mathbb{C}_q(\mathcal{K})$ we obtain the irreducible decomposition

$$\mathbb{C}_q^\sigma(\mathcal{K}/\mathcal{M}) \simeq \bigoplus_{\lambda \in \Lambda_n} V_\lambda^q$$

as $\mathcal{U}_q$-modules. Furthermore, for $\mu \in P_d^+$ we recall from [5] that

$$\left(L_\mu^q\right)^{\mathfrak{k}_\infty} = \left(L_\mu^q\right)^{\mathcal{U}_q(\mathfrak{m}_{\mathbb{C}})},$$

cf. Subsection 5.2.

We fix generic $\sigma, \tau \in \mathbb{R}$ unless specified differently (the case $\sigma = \infty$ and/or $\tau = \infty$ will subsequently be analyzed as degeneration of the theory for $\sigma, \tau \in \mathbb{R}$). For $\lambda \in \Lambda_n$ we choose a nonzero $\mathfrak{k}_\sigma$-fixed vector $v_\sigma(\lambda) \in \left(V_\lambda^q\right)^{\mathfrak{k}_\sigma}$ which we normalize by requiring

$$v_\sigma(\lambda) = \sum_{\nu \in \mathbb{Z}^d} v_\sigma(\lambda; \nu), \qquad v_\sigma(\lambda; \lambda^\natural) = u_{\lambda^\natural},$$

where $v_\sigma(\lambda; \nu) \in V_\lambda^q$ is the weight ν component of $v_\sigma(\lambda)$ in V_λ^q and $u_{\lambda^\natural}$ is the highest weight vector of $V_\lambda^q = L_{\lambda^\natural}^q$ (it is known that $v_\sigma(\lambda)$ has a weight $\lambda^\natural$ component, hence such a normalization is possible). From the analysis in [5, Section 6] it furthermore follows that $\left(V_\lambda^q\right)^{\mathfrak{k}_\sigma^*}$ is spanned by $K^{-\delta} v_{\sigma+d-2n}(\lambda)$ for $\lambda \in \Lambda_n$ (see also [22] for the special case $d = 2n$).

The normalized quantum zonal spherical functions $\varphi_\lambda^{\sigma,\tau}$ ($\sigma, \tau \in \mathbb{R}$) associated to the one-parameter family of quantum Grassmannians are now defined by

$$(8.1) \qquad \varphi_\lambda^{\sigma,\tau}(\,\cdot\,) = q^{\langle \delta, \lambda^\natural \rangle} \frac{\langle\,\cdot\, v_\sigma(\lambda), K^{-\delta} v_{\tau+d-2n}(\lambda)\rangle_\lambda}{\langle u_{\lambda^\natural}, u_{\lambda^\natural} \rangle_\lambda}, \qquad \lambda \in \Lambda_n.$$

The $\varphi_\lambda^{\sigma,\tau}$ ($\lambda \in \Lambda_n$) form a basis of

$$\mathcal{H}_{\sigma,\tau} := \{f \in \mathbb{C}_q(\mathcal{K}) \mid f(YZ) = 0 = f(ZX), \quad Z \in \mathcal{U}_q,\ X \in \mathfrak{k}_\sigma,\ Y \in \mathfrak{k}_\tau\}.$$

The analogue of the quantum dimension formula (5.4) in terms of the quantum zonal spherical functions $\varphi_\lambda^{\sigma,\tau}$ ($\lambda \in \Lambda_n$) reads as follows.

Lemma 8.1. *For $\lambda \in \Lambda_n$ and generic $\sigma, \tau \in \mathbb{R}$ we have*

$$(8.2) \qquad \operatorname{Dim}_q\big(V_\lambda^q\big) = \frac{\varphi_\lambda^{\sigma,\sigma+2n-d}(K^\delta)\varphi_\lambda^{\tau+d-2n,\tau}(K^\delta)}{\|\varphi_\lambda^{\sigma,\tau}\|_h^2}.$$

Proof. By the quantum Schur orthogonality relations (see Proposition 3.3**b**) we have

$$(8.3) \qquad \|\varphi_\lambda^{\sigma,\tau}\|_h^2 = \frac{q^{2\langle\delta,\lambda^\natural\rangle}}{\operatorname{Dim}_q(V_\lambda^q)} \frac{\|v_\sigma(\lambda)\|_\lambda^2 \|v_{\tau+d-2n}(\lambda)\|_\lambda^2}{\|u_{\lambda^\natural}\|_\lambda^4},$$

where $\|v\|_\lambda^2 := \langle v, v\rangle_\lambda$. On the other hand, from the definition (8.1) of the quantum zonal spherical function $\varphi_\lambda^{\sigma,\tau}$ we have

$$(8.4) \qquad \varphi_\lambda^{\sigma,\sigma+2n-d}(K^\delta) = \frac{\|v_\sigma(\lambda)\|_\lambda^2}{\|u_{\lambda^\natural}\|_\lambda^2} q^{\langle\delta,\lambda^\natural\rangle}.$$

The desired quantum dimension formula follows directly from (8.3) and (8.4). $\qquad\square$

To arrive at an explicit evaluation of the quantum dimensions, we now translate (8.2) into an expression involving the evaluations and the quadratic norms of Macdonald-Koornwinder polynomials.

The radial part $P_\lambda^{\sigma,\tau} := \operatorname{Res}_T\big(\varphi_\lambda^{\sigma,\tau}\big) \in \mathbb{C}[u^{\pm 1}]$ is defined to be the unique Laurent polynomial in d independent variables $u_1, \ldots, u_d$ such that

$$P_\lambda^{\sigma,\tau}(q^\nu) = \varphi_\lambda^{\sigma,\tau}(K^\nu) \qquad \forall \nu \in \mathbb{Z}^d.$$

We have the following key facts from [20], see also [5] and [22]. Set $z_j = u_j u_{d+1-j}^{-1}$ ($j = 1, \ldots, n$), then

$$P_\lambda^{\sigma,\tau} \in \mathbb{C}[z^{\pm 1}]^W, \qquad \forall \lambda \in \Lambda_n,$$

where $W = S_n \ltimes \{\pm 1\}^n$ as before. From now on we will view $P_\lambda^{\sigma,\tau}$ as W-invariant Laurent polynomial in the independent variables $z_1, \ldots, z_n$. The key result is the identification of $P_\lambda^{\sigma,\tau}$ with Macdonald-Koornwinder polynomials,

$$P_\lambda^{\sigma,\tau}(z) = P_\lambda(z; -q^{\sigma+\tau+1}, -q^{-\sigma-\tau+1}, q^{\sigma-\tau+1}, q^{-\sigma+\tau+2(d-2n)+1}; q^2, q^2), \qquad \lambda \in \Lambda_n$$

(the particular normalization of $\varphi_\lambda^{\sigma,\tau}$ ensures that it precisely coincides with the monic Macdonald-Koornwinder polynomial). We denote $\langle\cdot,\cdot\rangle_{\sigma,\tau}$ for the corresponding orthogonality pairing (6.3), with the six parameters (a, b, c, d, q, t) specialized to

$$(-q^{\sigma+\tau+1}, -q^{-\sigma-\tau+1}, q^{\sigma-\tau+1}, q^{-\sigma+\tau+2(d-2n)+1}, q^2, q^2).$$

We set $\|p\|_{\sigma,\tau}^2 = \langle p, p\rangle_{\sigma,\tau}^2$ for the corresponding quadratic norm of $p \in \mathbb{C}[z^{\pm 1}]^W$.

Lemma 8.2. *For $\lambda \in \Lambda_n$ and generic $\sigma, \tau \in \mathbb{R}$ we have*

$$\varphi_\lambda^{\sigma,\tau}(K^\delta) = P_\lambda^{\sigma,\tau}(q^{d-2n+1}q^{2\rho}),$$

$$\|\varphi_\lambda^{\sigma,\tau}\|_h^2 = \|P_\lambda^{\sigma,\tau}\|_{\sigma,\tau}^2.$$

In particular,

$$(8.5) \qquad Dim_q\big(V_\lambda^q\big) = \frac{P_\lambda^{\sigma,\sigma+2n-d}(q^{d-2n+1}q^{2\rho})P_\lambda^{\tau+d-2n,\tau}(q^{d-2n+1}q^{2\rho})}{\|P_\lambda^{\sigma,\tau}\|_{\sigma,\tau}^2}.$$

Proof. Observe that $u = q^\delta \in \left(\mathbb{R}^\times\right)^d$ maps to $z = q^{d-2n+1}q^{2\rho} \in \left(\mathbb{R}^\times\right)^n$ under the assignment $z_j = u_j u_{d+1-j}^{-1}$ $(j = 1, \ldots, n)$. By the definition of $P_\lambda^{\sigma,\tau}$ as the radial part of $\varphi_\lambda^{\sigma,\tau}$, we obtain the desired expression of $\varphi_\lambda^{\sigma,\tau}(K^\delta)$ in terms of Macdonald-Koornwinder polynomials. The quadratic norm formula follows from [5, Corollary 8.1]. The quantum dimension formula (8.2) then yields (8.5). □

We are now in a position to derive the explicit quantum dimension formulas for the spherical $\mathcal{U}_q$-representations V_λ^q ($\lambda \in \Lambda_n$).

Corollary 8.3. *Proposition 5.3 holds.*

Proof. Observe that the value of the Macdonald-Koornwinder polynomial

$$P_\lambda^{\sigma,\sigma+2n-d}(z) = P_\lambda(z; -q^{2\sigma+2n-d+1}, -q^{-2\sigma-2n+d+1}, q^{d-2n+1}, q^{d-2n+1}; q^2, q^2),$$

at $z = q^{d-2n+1}q^{2\rho}$ can be evaluated in closed form by the evaluation formula (6.2) for Macdonald-Koornwinder polynomials. The resulting expression is

$$P_\lambda^{\sigma,\sigma+2n-d}(q^{d-2n+1}q^{2\rho}) = q^{(2n-d-1)|\lambda|-2\langle\rho,\lambda\rangle}$$

$$\times \prod_{1 \le j < k \le n} \frac{(1 - q^{2(d-2n+1+\lambda_j+\lambda_k+\rho_j+\rho_k)})(1 - q^{2(\lambda_j-\lambda_k+\rho_j-\rho_k)})}{(1 - q^{2(d-2n+1+\rho_j+\rho_k)})(1 - q^{2(\rho_j-\rho_k)})}$$

$$\times \prod_{i=1}^n \frac{\left(q^{2(d-2n+1+\rho_i)}, q^{2(d-2n+1+\rho_i)}, -q^{2(\sigma+1+\rho_i)}, -q^{2(-\sigma+d-2n+1+\rho_i)}; q^2\right)_{\lambda_i}}{\left(q^{2(d-2n+1+2\rho_i)}; q^2\right)_{2\lambda_i}}.$$

On the other hand, by the closed expression (6.4) for the quadratic norms of the Macdonald-Koornwinder polynomials we have

$$\|P_\lambda^{\sigma,\tau}\|_{\sigma,\tau}^2 = \prod_{j=1}^n \left\{ \frac{\left(q^{2(1+\rho_j)}, q^{2(d-2n+1+\rho_j)}; q^2\right)_{\lambda_j}^2}{\left(q^{2(d-2n+1+2\rho_j)}, q^{2(d-2n+2+2\rho_j)}; q^2\right)_{2\lambda_j}} \right.$$

$$\left. \times \left(-q^{2(\sigma+1+\rho_j)}, -q^{2(-\sigma+d-2n+1+\rho_j)}, -q^{2(\tau+d-2n+1+\rho_j)}, -q^{2(-\tau+1+\rho_j)}; q^2\right)_{\lambda_j} \right\}.$$

Combining these explicit formulas with (8.5) we obtain an explicit expression for $\mathrm{Dim}_q(V_\lambda^q)$, which results in Proposition 5.3. □

We end this section by considering the $\sigma \to \infty$ and $\sigma = \tau \to \infty$ degenerations in the above representation theoretic formulas. This in particular leads to the interpretation of $D_q(\lambda; a, b; t)$ for special values of the parameters as quantum dimensions of V_λ^q ($\lambda \in \Lambda_n$), see (5.6).

In [5] the limit $\sigma \to \infty$ (respectively $\sigma, \tau \to \infty$) is considered, leading to the interpretation of the multivariable big q-Jacobi polynomials $P_\lambda^B(\cdot; 1, q^{2(d-2n)}, 1, q^{2\tau+2(d-2n)}; q^2, q^2)$ (respectively the multivariable little q-Jacobi polynomials $P_\lambda^L(\cdot; q^{2(d-2n)}, 1; q^2, q^2)$) as radial parts of quantum zonal spherical functions associated to the standard quantum Grassmann representation $\mathbb{C}_q^\infty(\mathcal{K}/\mathcal{M}) = \mathbb{C}_q(\mathcal{K}/\mathcal{M})$. In these degenerations, the quantum dimensions $\mathrm{Dim}_q(V_\lambda^q)$ are naturally expressed in terms of the multivariable big and little q-Jacobi polynomials as follows.

Proposition 8.4. *We have*

$$\operatorname{Dim}_q(V_\lambda^q) = \left(-q^{2\tau+d-2n-1}\right)^{|\lambda|} \frac{P_\lambda^L(0; q^{2(d-2n)}, 1; q^2, q^2) P_\lambda^{\tau+d-2n,\tau}(q^{d-2n+1}q^{2\rho})}{N_B(\lambda; 1, q^{2(d-2n)}, 1, q^{2\tau+2(d-2n)}; q^2, q^2)}, \qquad \forall\, \lambda \in \Lambda_n.$$

Furthermore, in terms of the generalized quantum dimension $D_q(\lambda; a, b; t)$, see (2.4), we have

$$\operatorname{Dim}_q(V_\lambda^q) = \frac{\left(P_\lambda^L(0; q^{2(d-2n)}, 1; q^2, q^2)\right)^2}{N_L(\lambda; q^{2(d-2n)}, 1; q^2, q^2)} = D_{q^2}(\lambda; q^{2(d-2n+1)}, q^2; q^2), \qquad \forall\, \lambda \in \Lambda_n.$$

Proof. We degenerate the expression (8.5) of the quantum dimension $\operatorname{Dim}_q(V_\lambda^q)$ by taking the limit $\sigma \to \infty$ (respectively $\sigma = \tau \to \infty$). In order to parallel the particular degeneration scheme from [5], we make use of the elementary symmetry

(8.6) $$P_\lambda(-z; -a, -b, -c, -d; q, t) = (-1)^{|\lambda|} P_\lambda(z; a, b, c, d; q, t)$$

for the monic Macdonald-Koornwinder polynomials, which implies the symmetry

(8.7) $$N_K(\lambda; -a, -b, -c, -d; q, t) = N_K(\lambda; a, b, c, d; q, t), \qquad \lambda \in \Lambda_n$$

for the corresponding quadratic norms (which also immediately follows from its explicit evaluation (6.2)). Combined with the limit formula (6.36) in base q^2, with parameters (a, b, c, d, t) specialized to $(1, q^{2(d-2n)}, 1, q^{2\tau+2(d-2n)}, q^2)$ and with $\epsilon = q^{\sigma+\tau-1}$, we obtain

$$\lim_{\sigma \to \infty} q^{2(\sigma+\tau-1)|\lambda|} \|P_\lambda^{\sigma,\tau}\|_{\sigma,\tau}^2 = N_B(\lambda; 1, q^{2(d-2n)}, 1, q^{2\tau+2(d-2n)}; q^2, q^2), \qquad \forall\, \lambda \in \Lambda_n.$$

On the other hand, from (6.11) and the explicit evaluation formula from the proof of Corollary 8.3 we obtain

$$\lim_{\sigma \to \infty} \left(-q^{2\sigma}\right)^{|\lambda|} P_\lambda^{\sigma,\sigma+2n-d}(q^{d-2n+1}q^{2\rho}) = q^{(d-2n+1)|\lambda|} P_\lambda^L(0; q^{2(d-2n)}, 1; q^2; q^2), \qquad \forall\, \lambda \in \Lambda_n.$$

Taking the limit $\sigma \to \infty$ in (8.5) now leads to the expression of the quantum dimension as mixture of factors involving Macdonald-Koornwinder, multivariable big and multivariable little q-Jacobi polynomials.

Taking $\tau = \sigma$ in (8.5) and using the fact that

$$\lim_{\sigma \to \infty} \left(q^{-1+2\sigma}\right)^{2|\lambda|} \|P_\lambda^{\sigma,\sigma}\|_{\sigma,\sigma}^2 = N_L(\lambda; q^{2(d-2n)}, 1; q^2, q^2), \qquad \forall\, \lambda \in \Lambda_n$$

in view of (6.21), we similarly obtain the desired expression of the quantum dimension $\operatorname{Dim}_q(V_\lambda^q)$ in terms of $D_q(\lambda; a, b; t)$. $\qquad\square$

Remark 8.5. By the arguments from [5] one can show that

$$\lim_{\sigma \to \infty} \left(\frac{\varphi_\lambda^{\sigma,\sigma+2n-d}}{\|\varphi_\lambda^{\sigma,\sigma+2n-d}\|_h}\right) = \varphi_\lambda^q, \qquad \lambda \in \Lambda_n$$

in the pre-Hilbert space $\mathbb{C}_q(\mathcal{K})$, where φ_λ^q ($\lambda \in \Lambda_n$) are the (orthonormal) quantum zonal spherical functions associated to the standard quantum complex Grassmanian. Degenerating (8.2) accordingly, we obtain the quantum dimension formula (5.4).

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